

Almgren-minimality of unions of two almost orthogonal planes in \mathbb{R}^4

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Abstract.

In this article we prove that the union of two almost orthogonal planes in \mathbb{R}^4 is Almgren-minimal. This gives an example of a one parameter family of minimal cones, which is a phenomenon that does not exist in \mathbb{R}^3 . This work is motivated by an attempt to classify the singularities of 2-dimensional Almgren-minimal sets in \mathbb{R}^4 . Note that the traditional methods for proving minimality (calibrations and slicing arguments) do not apply here, we are obliged to use some more complicated arguments such as a stopping time argument, harmonic extensions, Federer-Fleming projections, etc. that are rarely used to prove minimality (they are often used to prove regularity). The regularity results for 2-dimensional Almgren minimal sets ([5],[6]) are also needed here.

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1 Introduction and preliminaries

1.1 Introduction

One of the main topics in geometric measure theory is the theory of minimal sets, currents and surfaces, which aims at important progress in understanding the regularity and existence of physical objects that have certain minimizing properties such as soap films. This is known in physics as Plateau's problem. Recall that a most simple version of Plateau's problem aims at finding a surface which minimizes area among all the surfaces having a given curve as boundary. See works of Besicovitch, Federer, Fleming, De Giorgi, Douglas, etc. for example.

Lots of notions of minimality have been introduced to modernize Plateau's problem, such as minimal surfaces, mass minimizing or size minimizing currents (see [18] for their definitions), varifold (c.f.[2]). In this article, we will mainly use the notion introduced by F.Almgren [3], in a general setting of sets, and which gives a very good description of the behavior of soap films. Note that soap films are

2-dimensional objects, but a general definition of d -dimensional minimal sets in an open set $U \subset \mathbb{R}^n$ is not more complicated.

Intuitively, a d -dimensional minimal set E in an open set $U \subset \mathbb{R}^n$ is a closed set E whose d -dimensional Hausdorff measure could not be decreased by any local Lipschitz deformation. (See Section 1.2 for the precise definition.)

The point of view here is very different from those of minimal surfaces and mass minimizing currents under certain boundary condition, which are more usually used. Comparing to the big number of results in the theory of mass minimizing currents, or classical minimal surface, in our case, very little results of regularity and existence are known. However, Plateau's problem is more like the study of size minimizing currents, for which much less results are known either (see [17] for certain existence results). One can prove that the support of a size minimizing current is automatically an Almgren minimizer, so that all the regularity results listed below are also true for supports of size minimizing currents.

First regularity results for minimal sets have been given by Frederick Almgren [3] (rectifiability, Ahlfors regularity in arbitrary dimension), then generalized by Guy David and Stephen Semmes [7] (uniform rectifiability, big pieces of Lipschitz graphs), Guy David [4] (minimality of the limit of a sequence of minimizers).

Since minimal sets are rectifiable and Ahlfors regular, they admit a tangent plane at almost every point. But our main interest is to study those points where there is no tangent plane, i.e. singular points.

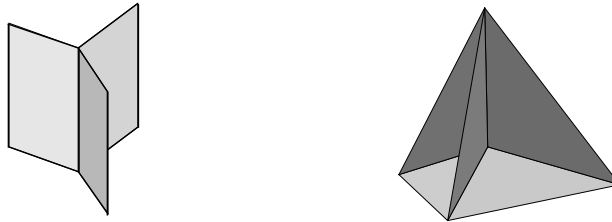
A first finer description of the interior regularity for minimal sets is due to Jean Taylor, who gave in [20] an essential regularity theorem for 2-dimensional minimal sets in 3-dimensional ambient spaces: if E is a minimal set of dimension 2 in an open set of \mathbb{R}^3 , then every point x of E has a neighborhood where E is equivalent (modulo a negligible set) through a C^1 diffeomorphism to a minimal cone (that is, a minimal set which is also a cone).

In [5], Guy David generalized Jean Taylor's theorem to 2-dimensional minimal sets in \mathbb{R}^n , but with a local bi-Hölder equivalence, that is, every point x of E has a neighborhood where E is equivalent through a bi-Hölder diffeomorphism to a minimal cone C (but the minimal cone might not be unique).

In addition, in [6], David also proved that, if this minimal cone C satisfies a "full-length" condition, we will have the C^1 equivalence (called C^1 regularity). In particular, the tangent cone of E at the point $x \in E$ exists and is a minimal cone, and the blow-up limit of E at x is unique; if the full-length condition fails, we might lose the C^1 regularity.

Thus, the study of singular points is transformed into the classification of singularities, i.e., into looking for a list of minimal cones. Besides, getting such a list would also help deciding locally what kind (i.e. C^1 or bi-Hölder) of equivalence with a minimal cone can we get.

In \mathbb{R}^3 , the list of 2-dimensional minimal cones has been given by several mathematicians a century ago. (See for example [11] or [10]). They are, modulo isomorphism: a plane, a \mathbb{Y} set (the union of 3 half planes that meet along a straight line where they make angles of 120 degrees), and a \mathbb{T} set (the cone over the 1-skeleton of a regular tetrahedron centered at the origin). See the pictures below.



In higher dimensions, even in dimension 4, the list of minimal cones is still very far from clear. Except for those three minimal cones that already exist in \mathbb{R}^3 , the only 2-dimensional minimal cone that was known before this paper is the union of two orthogonal planes, whose minimality could be proved by a simple projection argument.

It was not even known whether this is the only minimal cone with this shape, i.e., as a union of two planes.

Note that in \mathbb{R}^3 , no small perturbation of any minimal cones ever preserves the minimality, and even, each minimal cone admits a different topology from the others. But in this article we will show the following theorem, which exhibits a phenomenon that does not exist in \mathbb{R}^3 :

Theorem 1.1. *There exists a continuous one-parameter family of 2-dimensional minimal cones \mathbb{R}^n for $n \geq 4$.*

In fact, we are going to prove the following more precise theorem, of which Theorem 1.1 is a direct but noticeable corollary.

Theorem 1.2 (minimality of the union of two almost orthogonal planes). *There exists $0 < \theta < \frac{\pi}{2}$, such that if P^1 and P^2 are two planes in \mathbb{R}^4 whose characteristic angles (α_1, α_2) satisfy $\alpha_2 \geq \alpha_1 \geq \theta$, then their union $P^1 \cup P^2$ is a minimal cone in \mathbb{R}^4 .*

The characteristic angles of two planes P^1, P^2 describe their relative position, the definition will be given in Section 2. But an equivalent statement of Theorem 1.2 that might be easier to understand is that any almost orthogonal union of two planes in \mathbb{R}^4 is minimal, i.e. there exists $a > 0$, such that if for any $v_1 \in P^1, v_2 \in P^2$,

$$(1.3) \quad | \langle v_1, v_2 \rangle | \leq a \|v_1\| \|v_2\|$$

then $P^1 \cup P^2$ is minimal. Or intuitively, there is an "open set" of unions of two planes, which contains the orthogonal union of two planes, such that each element in this set (which is a union of two planes) is a minimal cone.

When a is small, (1.3) is almost equivalent to saying that $\alpha_1, \alpha_2 > \frac{\pi}{2} - Ca$. Hence we really have a one parameter family of minimal cones (which are not C^1 equivalent), whose intersections with the unit ball have the same Hausdorff measure.

Moreover, it is not hard to check that each cone in this one parameter family verifies the "full-length property", hence by Thm 1.15 of [6], if a blow-up limit of a minimal set E at a point x is one of these unions of planes P_α , then locally E is C^1 equivalent to P_α .

This is also a general question for unions of higher dimensional planes. Unions of orthogonal planes are always minimal, but for non-orthogonal cases we know very little. Some new information can be given, using the arguments and results of this paper and the notion of topological minimal sets, see [14].

For the proof of the theorem, the argument is not simply a perturbation analysis. Due to the higher codimension and the lack of results for minimal sets in \mathbb{R}^4 , the classical methods (such as calibration, slicing) for proving minimality do not apply, hence we are obliged to use tools in harmonic analysis, which are rarely used to prove minimality results. In particular, we use a stopping time argument to divide minimal sets into good and bad parts, and we treat them differently (projection for the bad part, harmonic extension for the good part), to get the estimates for their Hausdorff measure. It would also be much simpler if we had the existence results for minimal sets, because this would allow us to do directly the stopping time argument. However we can only use a partial existence result of [9]. We use it to get some sort of minimal set, and we have to prove some necessary regularity for it, so that the stopping time argument can be carried on well. Some of the arguments, especially the geometric constructions, are somehow painful. But we have not found any other easier way to do it.

Remark 1.4. 1) *To the author's knowledge, the only proof of minimality that does not use a calibration argument (probably not available here) is the proof of minimality for the cone T , suggested by J. Taylor in [20], using a topological argument, and the list of all the other minimal cones. But in our case the topology is more complicated, existence results are weaker, and moreover the list of the other minimal cones are far from known.*

2) *Frank Morgan gave a conjecture in [16] on the angle condition (which is imposed on the sum of the two characteristic angles) under which a union of two planes is minimal. It was proved by Gary Lawlor [12] that if the sum of their two characteristic angles is less than $\frac{2\pi}{3}$, then the union of the two planes will not be minimal. The remaining part is still open.*

Our theorem also solves partially the remaining part of this conjecture, because if $\alpha_1 + \alpha_2 > \frac{\pi}{2} + \theta$ (θ is as in Theorem 1.2), then we'll have $\alpha_2 \geq \alpha_1 > \theta$ (since both of them are no more than $\pi/2$), and Theorem 1.2 implies that the union of two planes with this pair of angles is minimal.

The strategy of the proof of Theorem 1.2 is the following.

Since our objects are cones centered at the origin, to prove that their measure could not be decreased by any deformation in any compact set in \mathbb{R}^n (which is the definition for minimal sets, see Subsection 1.1 for the precise definition), we can just look at deformations in the unit ball $B = B(0, 1)$. For $\theta = (\theta_1, \theta_2)$, denote by P_θ the union of two planes with characteristic angles θ . We begin by verifying that $P_{(\frac{\pi}{2}, \frac{\pi}{2})}$ is minimal, and moreover is the only minimal set in the unit ball among all sets with the same boundary and surjective orthogonal projections on the two planes. Here the boundary of a set is just the intersection of its closure with the unit sphere.

Next we begin to prove Theorem 1.2. Suppose the conclusion of the theorem is false. Then there exists a sequence $\{\theta_k\}$ which converges to $\theta_0 := (\frac{\pi}{2}, \frac{\pi}{2})$, such that P_{θ_k} is not minimal. Set $P_k = P_{\theta_k}$, and $P_0 = P_0^1 \cup_{\theta_0} P_0^2 = P_0^1 \cup_{\perp} P_0^2$. So we have

$$(1.5) \quad \inf\{H^2(F) : F \subset \overline{B}(0, 1) \text{ is a deformation of } P_k \text{ in } B\} < H^2(P_k \cap B) = 2\pi.$$

The next step would be easier if we could find, for each k , a deformation E_k of P_k in B which minimizes $H^2(E \cap B)$ among all deformations E of P_k in B . Unfortunately, no such existence theorem is known.

However, Vincent Feuvrier showed in his thesis [9] a partial existence result, which says that given an original set E in \overline{B} , there exists a certain set F that is minimal in B , which is the limit of a minimizing sequence of deformations of E in B , and such that

$$(1.6) \quad H^2(F) \leq \inf\{H^2(E') : E' \subset \overline{B}(0, 1) \text{ is a deformation of } E \text{ in } B(0, 1)\}.$$

We cannot say that this set F obtained by limit of deformations is still a deformation of E , but luckily we shall not need to know that. We show first that for $i = 1, 2$, the projection of each set E_k on the plane P_k^i contains $P_k^i \cap B$; then we manage to show that E_k has the same boundary as P_k . We can also get

$$(1.7) \quad H^2(E_k) < 2\pi = H^2(P_k \cap B).$$

Hence we get our favorite sequence of minimal sets E_k , whose boundaries converge to the boundary $P_0 \cap \partial B$ of P_0 (recall that $P_0 = P_0^1 \cup_{\perp} P_0^2$). We take a convergent subsequence, that we still denote by $\{E_k\}$. By [4] Thm 4.1, the limit of a sequence of minimal sets is still minimal, hence the limit E_∞ of $\{E_k\}$ is minimal. Now $E_\infty \cap \partial B = P_0 \cap \partial B$, hence the uniqueness theorem for P_0 tells us that E_∞ cannot be anyone other than P_0 . Moreover, the Hausdorff distance between E_k and P_k also converges to 0, because P_k converges also to P_0 . See Section 4 for detail.

Now for every fixed small ϵ , we will use a stopping time argument to decompose every E_k . More precisely, we use a stopping time argument to find, for each k , a critical radius $r_k = r_k(\epsilon)$, such that E_k is ϵ -near P_k or a translation outside a ball $B(o_k, r_k)$, with o_k fairly near the origin. Moreover in $B(o_k, r_k)$, E_k begins to get away a little from P_k , that is, r_k is the scale in which E_k begins to go away

from P_k , even though we do not understand how. In spite of that, we will manage to show that the projections of the part of E_k inside $B(o_k, r_k)$ on the two planes are surjective. (The proof of this projection property is essentially derived from the construction in the proof of the partial existence result of [9], which is somehow complicated).

We know that r_k is the very first radius such that E_k is not ϵr_k near P_k , so when ϵ is sufficiently small, by the regularity of minimal sets near flat points, we manage to show that outside the small ball $B(o_k, \frac{1}{4}r_k)$, E_k is the union of two C^1 graphs on P_k^1, P_k^2 respectively. Then we decompose E_k into two parts, which are outside and inside $B(o_k, \frac{1}{4}r_k)$ respectively.

For the part outside, we already know that E_k is composed of two graphs G_i of C^1 functions f_i from the annuli $P_k^i \cap B(0, 1) \setminus B(o_k, \frac{1}{4}r_k)$ to $P_k^{i\perp}$ respectively. But the union of two parts is not ϵr_k -near any translation of P_k in $B(o_k, r_k) \setminus B(o_k, \frac{1}{4}r_k)$ (in fact, we have only shown that E_k is not ϵr_k -near any translation of P_k in $B(o_k, r_k)$ rather than $B(o_k, r_k) \setminus B(o_k, \frac{1}{4}r_k)$, but then we can prove by a compactness argument that E_k is not too close to any translation of P_k in $B(o_k, r_k) \setminus B(o_k, \frac{1}{4}r_k)$ either, maybe at the price of making ϵ smaller. See Section 8 for more detail). Thus, at least one of the two graphs is $\frac{1}{2}\epsilon r_k$ far from any translation of its domain. Suppose this is the case for $i = 1$. This means that f_1 oscillates of order ϵr_k in the region $B(o_k, r_k) \setminus B(o_k, \frac{1}{4}r_k)$. Then some argument using harmonic functions gives that the Dirichlet energy $\int_{P_k^1 \cap B(0, 1) \setminus B(o_k, \frac{1}{4}r_k)} |\nabla f_1|^2$ of f_1 on $P_k^1 \cap B(0, 1) \setminus B(o_k, \frac{1}{4}r_k)$ is larger than $C(\epsilon)r_k^2$. But this is almost equivalent to the difference between the measure of the graph G_1 and the measure of the annulus $P_k^1 \cap B(0, 1) \setminus B(o_k, \frac{1}{4}r_k)$, so we have

$$(1.8) \quad H^2(E_k \cap B(0, 1) \setminus B(o_k, \frac{1}{4}r_k)) \geq H^2(P_k^1 \cap B(0, 1) \setminus B(o_k, \frac{1}{4}r_k)) + C(\epsilon)r_k^2.$$

This means we gain some measure $C(\epsilon)r_k^2$, where the constant $C(\epsilon)$ does not depend on k .

For the inside part, by a projection argument we can show that

$$(1.9) \quad \begin{aligned} H^2(E_k \cap B(o_k, \frac{1}{4}r_k)) &\geq (1 - C \times (\frac{\pi}{2} - \theta_k))H^2(P_k \cap B(o_k, \frac{1}{4}r_k)) \\ &= H^2(P_k \cap B(o_k, \frac{1}{4}r_k)) - C \times (\frac{\pi}{2} - \theta_k)r_k^2, \end{aligned}$$

where (recall that) $\frac{\pi}{2} - \theta_k \rightarrow 0$ as $k \rightarrow \infty$.

Altogether we get

$$(1.10) \quad H^2(E_k) \geq H^2(P_k \cap B(0, 1)) + [C(\epsilon) - C \times (\frac{\pi}{2} - \theta_k)]r_k^2.$$

When $k \rightarrow \infty$, the term $[C(\epsilon) - C \times (\frac{\pi}{2} - \theta_k)]r_k^2$ is strictly positive, so the measure of $E_k \cap B$ is strictly larger than 2π . But this cannot happen, because of (1.7).

Thus we finish the proof of Theorem 1.2.

Here we have to point out that if we could control how fast are the E_k converging to P_0 , then we would be able to give an estimate on the term $C(\epsilon) - C \times (\frac{\pi}{2} - \theta_k)$, and thus to give an estimate on

the angle θ in Theorem 1.2. But such a control might need a deeper understanding of the uniqueness theorem of P_0 (Thm 3.1).

The same kind of argument for proving Theorem 1.2 can be generalized (but not trivially) to prove the minimality of the union of n almost orthogonal planes of dimension m in \mathbb{R}^{mn} (which are also related to Morgan's conjecture), and also the union of a plane and a \mathbb{Y} in \mathbb{R}^5 . See [13] for detail.

The plan for the rest of this article is the following.

Subsection 1.1 will give some notation and conventions that we will use frequently afterwards.

Section 2 is devoted to the estimation of the sum of projections of a simple unit 2-vector to two planes, estimation that depends on the characteristic angles of the two planes. Based on this we give a comparison between the measure of a rectifiable set and the sum of the measures of its projections on the two planes.

In Section 3 we prove that P_0 is the only minimal set in $B(0,1)$ with the given boundary and surjective projections.

In Section 4 we show the existence of the minimal sets E_k , as well as some of their properties.

Section 5 is devoted to finding the critical radius r_k , and giving some properties of E_k outside the "ball" $D(o_k, \frac{1}{4}r_k)$.

In Section 6 we prove that the projections of E_k on P_k^1 and P_k^1 are surjective, in $B(0,1)$, as well as in $B(o_k, t)$ for all $t \in [\frac{1}{4}r_k, r_k]$. We also give the C^1 regularity of E_k outside the ball $B(o_k, \frac{1}{4}r_k)$.

Section 7 contains an argument of harmonic extension, which gives a lower bound for the measure of the graph of a function, depending on the size of the order of its oscillations near the boundary.

Finally we arrive at our conclusion in Section 8, by combining all the information we gathered before.

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1.2 Preliminaries

- In all that follows, minimal set means Almgren minimal set;
- $[a, b]$ is the line segment with end points a and b ;
- $[a, b)$ is the half line issued from the point a and passing through b ;
- $B(x, r)$ is the open ball with radius r and centered on x ;
- $\overline{B}(x, r)$ is the closed ball with radius r and center x ;
- For any set $E \subset \mathbb{R}^n$, and $r > 0$, $B(E, r) := \{x \in \mathbb{R}^n, d(x, E) < r\} = \cup_{x \in E} B(x, r)$;
- \overrightarrow{ab} is the vector $b - a$;

- H^d is the Hausdorff measure of dimension d ;
- $d_H(E, F) = \max\{\sup\{d(y, F) : y \in E, \sup\{d(y, E) : y \in F\}\}$ is the Hausdorff distance between two sets E and F .

In the next definitions, fix integers $0 < d < n$. We are mostly interested in $d = 2$ and $n = 4$ here.

Definition 1.11 (Almgren competitor (Al competitor for short)). *Let E be a closed set in an open subset U of \mathbb{R}^n . An Almgren competitor for E is a closed set $F \subset \overline{U}$ that can be written as $F = \varphi_1(E)$, where $\varphi_t : U \rightarrow U$ is a family of continuous mappings such that*

$$(1.12) \quad \varphi_0(x) = x \text{ for } x \in U;$$

$$(1.13) \quad \text{the mapping } (t, x) \rightarrow \varphi_t(x) \text{ of } [0, 1] \times U \text{ to } U \text{ is continuous};$$

$$(1.14) \quad \varphi_1 \text{ is Lipschitz,}$$

and if we set $W_t = \{x \in U ; \varphi_t(x) \neq x\}$ and $\widehat{W} = \bigcup_{t \in [0, 1]} [W_t \cup \varphi_t(W_t)]$, then

$$(1.15) \quad \widehat{W} \text{ is relatively compact in } U.$$

Such a φ_1 is called a deformation in U , and F is also called a deformation of E in U .

Definition 1.16 ((Almgren) minimal sets). *Let $0 < d < n$ be integers, U an open set of \mathbb{R}^n . A closed set E in U is said to be minimal of dimension d in U if*

$$(1.17) \quad H^d(E \cap B) < \infty \text{ for every compact ball } B \subset U,$$

and

$$(1.18) \quad H^d(E \setminus F) \leq H^d(F \setminus E)$$

for all Al competitors F for E .

Remark 1.19. *When the ambient set U is \mathbb{R}^n , or a ball, we can also take the class of local Almgren competitors to define the same notion of minimal set. Keep the E , U , n and d as before; a local Almgren competitor of E in U is a set $F = f(E)$, with*

$$(1.20) \quad f = \text{id outside some compact ball } B \subset U,$$

$$(1.21) \quad f(B) \subset B,$$

and f is Lipschitz.

A such f is called a local deformation in U , or a deformation in B , and $F = f(E)$ is also called a local deformation of E in U , or a deformation of E in B .

Note that in this case, the condition (1.18) becomes

$$(1.22) \quad H^d(E \cap B) \leq H^d(F \cap B).$$

We say that a set E closed in an open set U is locally minimal if (1.17) holds, and for any compact ball $B \subset U$, and any local Almgren competitor F for E in B , (1.22) holds.

One can easily verify that when U is \mathbb{R}^n or a ball, the class of Al competitors coincides with the class of local Al competitors, so the two classes define the same kind of minimal sets. However, if the ambient set U has a more complicated geometry, then the class of local Al competitors is strictly smaller, so a set minimizing the Hausdorff measure among local Al competitors might fail to be Al-minimal.

Remark 1.23. The notion of minimal sets does not depend much on the ambient dimension. One can easily check that $E \subset U$ is d -dimensional minimal in $U \subset \mathbb{R}^n$ if and only if E is minimal in $U \times \mathbb{R}^m \subset \mathbb{R}^{m+n}$, for any integer m .

Remark 1.24. For a description of a little more general kind of sets, such as soap bubbles (where the pressures are different in different connected components of their complementary), or taking gravity into consideration, we can use a notion of almost minimal sets, where we add some error term in the condition (1.18). To these sets, almost all the argument for minimal sets can be applied, and we have essentially the same regularity results. See [5] for definitions and more detail.

Definition 1.25 (reduced set). Let $U \subset \mathbb{R}^n$ be an open set. For every closed subset E of U , denote by

$$(1.26) \quad E^* = \{x \in E ; H^d(E \cap B(x, r)) > 0 \text{ for all } r > 0\}$$

the closed support (in U) of the restriction of H^d to E . We say that E is reduced if $E = E^*$.

It is easy to see that

$$(1.27) \quad H^d(E \setminus E^*) = 0.$$

In fact we can cover $E \setminus E^*$ by countably many balls B_j such that $H^d(E \cap B_j) = 0$.

Remark 1.28. If E is minimal, then E^* is also minimal, because if φ_1 is a deformation in U as defined in Definition 1.11, then it is Lipschitz, hence $H^d(\varphi(E \setminus E^*)) = H^d(E \setminus E^*) = 0$. So the condition (1.18) is the same for E^* as for E . As a result it is enough to study reduced minimal sets.

Definition 1.29 (blow-up limit). *Let $U \subset \mathbb{R}^n$ be an open set, let E be a relatively closed set in U , and let $x \in E$. Denote by $E(r, x) = r^{-1}(E - x)$. A set C is said to be a blow-up limit of E at x if there exists a sequence of numbers r_n , with $\lim_{n \rightarrow \infty} r_n = 0$, such that the sequence of sets $E(r_n, x)$ converges to C for the Hausdorff distance in any compact set of \mathbb{R}^n .*

Remark 1.30. *A set E might have more than one blow-up limit at a point x .*

We now state some regularity results that will be used throughout this paper.

Definition 1.31 (bi-Hölder ball for closed sets). *Let E be a closed set of Hausdorff dimension 2 in \mathbb{R}^n . We say that $B(0, 1)$ is a bi-Hölder ball for E , with constant $\tau \in (0, 1)$, if we can find a 2-dimensional minimal cone Z in \mathbb{R}^n centered at 0, and $f : B(0, 2) \rightarrow \mathbb{R}^n$ with the following properties:*

- 1° $f(0) = 0$ and $|f(x) - x| \leq \tau$ for $x \in B(0, 2)$;
- 2° $(1 - \tau)|x - y|^{1+\tau} \leq |f(x) - f(y)| \leq (1 + \tau)|x - y|^{1-\tau}$ for $x, y \in B(0, 2)$;
- 3° $B(0, 2 - \tau) \subset f(B(0, 2))$;
- 4° $E \cap B(0, 2 - \tau) \subset f(Z \cap B(0, 2)) \subset E$.

We also say that $B(0, 1)$ is of type Z .

We say that $B(x, r)$ is a bi-Hölder ball for E of type Z (with the same parameters) when $B(0, 1)$ is a bi-Hölder ball of type Z for $r^{-1}(E - x)$.

Theorem 1.32 (Bi-Hölder regularity for 2-dimensional minimal sets, c.f. [5] Thm 16.1). *Let U be an open set in \mathbb{R}^n and E a reduced minimal set in U . Then for each $x_0 \in E$ and every choice of $\tau \in (0, 1)$, there is an $r_0 > 0$ and a minimal cone Z such that $B(x_0, r_0)$ is a bi-Hölder ball of type Z for E , with constant τ . Moreover, Z is a blow-up limit of E at x .*

Definition 1.33 (point of type Z). *In the above theorem, we say that x_0 is a point of type Z (or Z point for short) of the minimal set E .*

Remark 1.34. *Again, since we might have more than one blow-up limit for a minimal set E at a point $x_0 \in E$, the point x_0 might have more than one type (but all blow-up limits at a point are bi-Hölder equivalent). However, if one of the blow-up limits of E at x_0 admits the "full-length" property (see Remark 1.36), then in fact E admits a unique blow-up limit at the point x_0 . Moreover, we have the following $C^{1,\alpha}$ regularity around the point x_0 . In particular, the blow-up limit of E at x_0 is in fact a tangent cone of E at x_0 .*

Theorem 1.35 ($C^{1,\alpha}$ -regularity for 2-dimensional minimal sets, c.f. [6] Thm 1.15). *Let E be a 2-dimensional reduced minimal set in the open set $U \subset \mathbb{R}^n$. Let $x \in E$ be given. Suppose in addition that some blow-up limit of E at x is a full length minimal cone (see Remark 1.36). Then there is a unique blow-up limit X of E at x , and $x + X$ is tangent to E at x . In addition, there is a radius $r_0 > 0$ such*

that, for $0 < r < r_0$, there is a $C^{1,\alpha}$ diffeomorphism (for some $\alpha > 0$) $\Phi : B(0, 2r) \rightarrow \Phi(B(0, 2r))$, such that $\Phi(0) = x$ and $|\Phi(y) - x - y| \leq 10^{-2}r$ for $y \in B(0, 2r)$, and $E \cap B(x, r) = \Phi(X) \cap B(x, r)$.

We can also ask that $D\Phi(x) = Id$. We call $B(x, r)$ a C^1 ball for E of type X .

Remark 1.36 (full length, union of two full length cones $X_1 \cup X_2$). We are not going to give the precise definition of the full length property. Instead, we just give some information here, which is enough for the proofs in this paper.

1) The three types of 2-dimensional minimal cones in \mathbb{R}^3 , i.e. the planes, the \mathbb{Y} sets, and the \mathbb{T} sets, all verify the full-length property (c.f., [6] Lemmas 14.4, 14.6 and 14.27). Hence all 2-dimensional minimal sets E in an open set $U \subset \mathbb{R}^3$ admits the local $C^{1,\alpha}$ regularity at every point $x \in E$. But this was known from [20].

2) (c.f., [6] Remark 14.40) Let $n > 3$. Note that the planes, the \mathbb{Y} sets and the \mathbb{T} sets are also minimal cones in \mathbb{R}^n . Denote by \mathfrak{C} the set of all planes, \mathbb{Y} sets and \mathbb{T} sets in \mathbb{R}^n . Let $X = \cup_{1 \leq i \leq n} X_i \in \mathbb{R}^n$ be a minimal cone, where $X_i \in \mathfrak{C}$, $1 \leq i \leq n$, and for any $i \neq j$, $X_i \cap X_j = \{0\}$. Then X also verifies the full-length property.

2 Projections on two orthogonal or almost orthogonal planes

In this section, we will give some estimates for the sum of the measures of projections of a rectifiable set on two transversal planes. These estimates are somewhat algebraic, and mainly use estimates for the sum of projections of simple unit 2-vectors in \mathbb{R}^4 . Here unit simple 2-vectors are used to represent planes and their relative positions, or are treated as surface elements or derivatives of functions between two rectifiable sets when we try to do some integration. As a result the orientation of a simple 2-vector will be ignored, in other words, we will essentially not need to distinguish between $x \wedge y$ and $y \wedge x$.

Denote by $\wedge_2(\mathbb{R}^4)$ the space of all 2-vectors in \mathbb{R}^4 . Let x, y be two vectors in \mathbb{R}^4 , we denote by $x \wedge y \in \wedge_2(\mathbb{R}^4)$ their exterior product. If $\{e_i\}_{1 \leq i \leq 4}$ is an orthonormal basis, then $\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}$ forms a basis of $\wedge_2(\mathbb{R}^4)$. We say that an element $v \in \wedge_2(\mathbb{R}^4)$ is simple if it can be expressed as the exterior product of two vectors.

The norm on $\wedge_2(\mathbb{R}^4)$, denoted by $|\cdot|$, is defined by

$$(2.1) \quad \left| \sum_{i < j} \lambda_{ij} e_i \wedge e_j \right| = \sum_{i < j} |\lambda_{ij}|^2.$$

Under this norm $\wedge_2(\mathbb{R}^4)$, is a Hilbert space, and $\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}$ is an orthonormal basis. For a simple 2-vector $x \wedge y$, its norm is

$$(2.2) \quad |x \wedge y| = \|x\| \|y\| \sin \angle_{x,y},$$

where $\angle_{x,y} \in [0, \pi]$ is the angle between the vectors x and y , and $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^4 . A unit simple 2-vector is a simple 2-vector of norm 1. Notice that $|\cdot|$ is generated by the scalar product \langle, \rangle defined on $\wedge_2(\mathbb{R}^4)$ as follows: for $\xi = \sum_{1 \leq i < j \leq 4} a_{ij} e_i \wedge e_j, \zeta = \sum_{1 \leq i < j \leq 4} b_{ij} e_i \wedge e_j$,

$$(2.3) \quad \langle \xi, \zeta \rangle = \sum_{1 \leq i < j \leq 4} a_{ij} b_{ij}.$$

One can easily verify that if two pairs of vectors x, y and x', y' generate the same 2-dimensional subspace of \mathbb{R}^4 , then there exists $r \in \mathbb{R} \setminus \{0\}$ such that $x \wedge y = r x' \wedge y'$.

Now given a unit simple 2-vector ξ , we can associate it to a 2-dimensional subspace $P(\xi) \in G(4, 2)$, where $G(4, 2)$ denotes the set of all 2-dimensional subspaces of \mathbb{R}^4 :

$$(2.4) \quad P(\xi) = \{v \in \mathbb{R}^4, v \wedge \xi = 0\}.$$

In other words, $P(x \wedge y)$ is the subspace generated by x and y .

From time to time, when there is no ambiguity, we write also $P = x \wedge y$, where $P \in G(4, 2)$ and the two unit vectors $x, y \in \mathbb{R}^4$ are such that $P = P(x \wedge y)$. In this case $x \wedge y$ represents a plane.

For the side of linear maps, if f is a linear map from \mathbb{R}^4 to \mathbb{R}^4 , then we denote by $\wedge_2 f$ (and sometimes f if there is no ambiguity) the linear map from $\wedge_2(\mathbb{R}^4)$ to $\wedge_2(\mathbb{R}^4)$ such that

$$(2.5) \quad \wedge_2 f(x \wedge y) = f(x) \wedge f(y).$$

And for the side of $G(4, 2)$ (the set of all planes, without considering orientations), for a unit simple 2-vector $\xi \in \wedge_2 \mathbb{R}^4$, we have always $P(\xi) = P(-\xi)$, so that we can define $|f(\cdot)| : G(4, 2) \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$$(2.6) \quad |f(P(\xi))| = |\wedge_2 f(\xi)|.$$

One can easily verify that the value of $|f(P(\xi))|$ does not depend on the choice of ξ that generates P . So $|f(\cdot)|$ is well defined.

2.1 Some estimate for the sum of projections of simple 2-vectors on two transversal planes

Let us recall the definition of characteristic angles between two planes. Let P^1, P^2 be two 2-dimensional planes in \mathbb{R}^4 . Among all pairs of unit vectors (v, w) with $v \in P^1, w \in P^2$, we choose (v_1, w_1) which minimizes the angle between them. We denote by α_1 this angle. Next we look at all the pairs of unit vectors $\{(v', w') : v' \in P^1, w' \in P^2, v' \perp v_1, w' \perp w_1\}$ (here P^1 and P^2 are 2-dimensional, so once w_1, v_1 are fixed, generally we only have four such pairs, $(\pm v_2, \pm w_2)$), and we choose (v_2, w_2) which minimize the angle among all such pairs. Denote by α_2 this angle. Then (α_1, α_2) (with $\alpha_1 \leq \alpha_2$) is the pair of characteristic angles between P^1 and P^2 .

Characteristic angles characterize absolutely the relative position between two planes, in the sense that we can find an orthonormal basis $\{e_i\}_{1 \leq i \leq 4}$ of \mathbb{R}^4 , such that

$$(2.7) \quad P^1 = e_1 \wedge e_2 \text{ and } P^2 = (\cos \alpha_1 e_1 + \sin \alpha_1 e_3) \wedge (\cos \alpha_2 e_2 + \sin \alpha_2 e_4).$$

Notice that two planes are orthogonal if their pair of characteristic angles is $(\frac{\pi}{2}, \frac{\pi}{2})$.

Now we want to estimate the sum of projections of a unit simple 2-vector on them. Denote by p^i the orthogonal projection on $P^i, i = 1, 2$. Then p^i is a linear map. We also denote by p^i the linear map $\wedge_2 p^i$ from $\wedge_2(\mathbb{R}^4)$ to itself, as defined in (2.5).

Now let ξ be a unit simple 2-vector, then there exists two unit vectors x, y such that $\xi = x \wedge y$ and $x \perp y$. We write x, y in the basis :

$$(2.8) \quad x = ae_1 + be_2 + ce_3 + de_4, y = a'e_1 + b'e_2 + c'e_3 + d'e_4$$

with

$$(2.9) \quad a^2 + b^2 + c^2 + d^2 = a'^2 + b'^2 + c'^2 + d'^2 = 1$$

and

$$(2.10) \quad aa' + bb' + cc' + dd' = 0.$$

We calculate the projections $|p^j(\xi)|$.

$$(2.11) \quad |p^1(\xi)| = | \langle e_1 \wedge e_2, \xi \rangle | = |ab' - a'b|,$$

and

$$(2.12) \quad \begin{aligned} |p^2(\xi)| &= | \langle (\cos \alpha_1 e_1 + \sin \alpha_1 e_3) \wedge (\cos \alpha_2 e_2 + \sin \alpha_2 e_4), \xi \rangle | \\ &= |(ab' - a'b) \cos \alpha_1 \cos \alpha_2 + (ad' - a'd) \cos \alpha_1 \sin \alpha_2 \\ &\quad + (cb' - c'b) \sin \alpha_1 \cos \alpha_2 + (cd' - c'd) \sin \alpha_1 \sin \alpha_2|. \end{aligned}$$

Then when $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, we have

$$(2.13) \quad \begin{aligned} |p^1(\xi)| + |p^2(\xi)| &= |ab' - a'b| + |cd' - c'd| \leq |ab'| + |a'b| + |cd'| + |c'd| \\ &\leq \frac{1}{2}(a^2 + b'^2 + a'^2 + b^2) + \frac{1}{2}(c^2 + d'^2 + c'^2 + d^2) = 1 \end{aligned}$$

because of (2.9). So we get the following lemma.

Lemma 2.14. *Let P^1, P^2 be two orthogonal planes, then for every unit simple 2-vector $\xi \in \wedge_2 \mathbb{R}^4$ we have*

$$(2.15) \quad |p^1(\xi)| + |p^2(\xi)| \leq 1.$$

More precisely, the next lemma gives exactly which are those unit simple 2-vectors satisfying equality in (2.15). Denote by Ξ the set of all unit simple 2-vectors $\xi \in \bigwedge_2 \mathbb{R}^4$ such that

$$(2.16) \quad |p^1(\xi)| + |p^2(\xi)| = 1.$$

Then $P(\Xi) = \{P(\xi), \xi \in \Xi\}$ is a compact subset of $G(4, 2)$ (c.f. [8], 1.6.2).

Lemma 2.17. *If $x \wedge y \in \Xi$, then there exists $\alpha \in [0, \frac{\pi}{2}]$, $v_i, u_i, i = 1, 2$ four unit vectors such that $v_i \in P^1$, $u_i \in P^2$, $v_1 \perp v_2, u_1 \perp u_2$, so that*

$$(2.18) \quad x = \cos \alpha v_1 + \sin \alpha u_1 \text{ and } y = \cos \alpha v_2 + \sin \alpha u_2.$$

Proof. This is just Wirtinger's inequality stated in 1.8.2 of [8], with $\nu = 2$, $\mathbb{R}^4 = \mathbb{C}_1 \oplus \mathbb{C}_2$, $P^1 = \mathbb{C}_1$, $P^{1\perp} = \mathbb{C}_2$, $\mu = 1$; the argument consists in checking the equality cases in (2.13). \square

Now we look at unions of two almost orthogonal planes.

Proposition 2.19. *Let $0 \leq \alpha_1 \leq \alpha_2 \leq \frac{\pi}{2}$, and let $P^1, P^2 \subset \mathbb{R}^4$ be two planes with characteristic angles $\alpha_1 \leq \alpha_2$. Denote by p^i the orthogonal projection on P^i , $i = 1, 2$. Then for any unit simple 2-vector $\zeta \in \bigwedge_2 \mathbb{R}^4$, its projections on these two planes satisfy:*

$$(2.20) \quad |p^1 \zeta| + |p^2 \zeta| \leq 1 + 2 \cos \alpha_1.$$

Remark 2.21. *Notice that when α_1 tends to $\frac{\pi}{2}$, $\cos \alpha_1$ tends to 0. So Proposition 2.19 implies that for ϵ small, there exists $\theta = \theta(\epsilon) \in]0, \frac{\pi}{2}[$ such that if $\alpha_2 \geq \alpha_1 \geq \theta$, then for all unit simple vectors $\zeta \in \bigwedge_2 \mathbb{R}^4$,*

$$(2.22) \quad |p^1 \zeta| + |p^2 \zeta| \leq 1 + \epsilon.$$

Proof of Proposition 2.19.

Let $0 \leq \alpha_1 \leq \alpha_2 \leq \frac{\pi}{2}$ be two arbitrary angles, and let P^1 and P^2 be a pair of planes with characteristic angles α_1, α_2 . Then there exists an orthonormal basis $\{e_i\}_{1 \leq i \leq 4}$ of \mathbb{R}^4 such that $P^1 = e_1 \wedge e_2$, $P^2 = (\cos \alpha_1 e_1 + \sin \alpha_1 e_3) \wedge (\cos \alpha_2 e_2 + \sin \alpha_2 e_4)$.

Denote by p the projection on the plane $P(e_3 \wedge e_4)$. For each unit simple $\zeta \in \bigwedge^2(\mathbb{R}^4)$, we have

$$(2.23) \quad |p^1 \zeta| + |p^2 \zeta| = |p^1 \zeta| + |p \zeta| + (|p^2 \zeta| - |p \zeta|) \leq |p^1 \zeta| + |p \zeta| + |p^2 \zeta - p \zeta|.$$

By Lemma 2.14,

$$(2.24) \quad |p^1 \zeta| + |p \zeta| \leq 1.$$

Hence we just need to estimate $|(p^2 - p)\zeta|$. By definition,

$$\begin{aligned}
(p^2 - p)(\zeta) &= \langle (\cos \alpha_1 e_1 + \sin \alpha_1 e_3) \wedge (\cos \alpha_2 e_2 + \sin \alpha_2 e_4) - e_3 \wedge e_4, \zeta \rangle \\
(2.25) \quad &= \langle \cos \alpha_1 \cos \alpha_2 e_1 \wedge e_2 + \cos \alpha_1 \sin \alpha_2 e_1 \wedge e_4 + \sin \alpha_1 \cos \alpha_2 e_3 \wedge e_2 \\
&\quad + (\sin \alpha_1 \sin \alpha_2 - 1) e_3 \wedge e_4, \zeta \rangle.
\end{aligned}$$

Notice that ζ is a unit 2-vector, so we have $\zeta = \sum_{1 \leq i < j \leq 4} a_{ij} e_i \wedge e_j$, where $\sum_{1 \leq i < j \leq 4} a_{ij}^2 = 1$. Hence

$$\begin{aligned}
&|(p^2 - p)(\zeta)| \\
&= |a_{12} \cos \alpha_1 \cos \alpha_2 + a_{14} \cos \alpha_1 \sin \alpha_2 - a_{23} \sin \alpha_1 \cos \alpha_2 + a_{34} (\sin \alpha_1 \sin \alpha_2 - 1)| \\
(2.26) \quad &\leq [\cos^2 \alpha_1 \cos^2 \alpha_2 + \cos^2 \alpha_1 \sin^2 \alpha_2 + \sin^2 \alpha_2 \cos^2 \alpha_2 + (\sin \alpha_1 \sin \alpha_2 - 1)^2]^{\frac{1}{2}} \\
&\quad \times [a_{12}^2 + a_{14}^2 + a_{23}^2 + a_{34}^2]^{\frac{1}{2}} \\
&\leq [\cos^2 \alpha_1 + \cos^2 \alpha_1 + \cos^2 \alpha_2 + (1 - \sin^2 \alpha_1)^2]^{\frac{1}{2}} \\
&\leq [3 \cos^2 \alpha_1 + \cos^4 \alpha_1]^{\frac{1}{2}} \leq [4 \cos^2 \alpha_1]^{\frac{1}{2}} = 2 \cos \alpha_1
\end{aligned}$$

by Cauchy-Schwarz and $\alpha_1 \leq \alpha_2$.

Combining (2.26), (2.23) and (2.24) we obtain the conclusion. \square

2.2 Comparison of the measure of a rectifiable set with the sum of measures of its projections on two planes

We will apply the estimates on simple 2-vectors obtained in the last subsection to rectifiable sets. Let F be a 2-rectifiable set. Then for almost all $x \in F$ the approximate tangent plane of F at x exists (c.f. [15] Thm 15.11), and we denote it by $T_x F$. Then $T_x F \in G(4, 2)$. For each linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $|f(T_x F)|$ is defined as in (2.6).

Lemma 2.27. *Let P^1, P^2 be two planes in \mathbb{R}^4 , let $F \subset \mathbb{R}^4$ be a 2-rectifiable set. Denote by p^i the projection on P^i . If λ is such that for almost all $x \in F$, the approximate tangent plane of F at x $T_x F \in G(4, 2)$ is such that*

$$(2.28) \quad |p^1(T_x F)| + |p^2(T_x F)| \leq \lambda,$$

then

$$(2.29) \quad H^2(p^1(F)) + H^2(p^2(F)) \leq \lambda H^2(F).$$

Proof.

Denote by f the restriction of p^1 on F , then f is a Lipschitz function from a 2-rectifiable set to a subset of a plane P^1 . Since F is 2-rectifiable, for H^2 -almost all $x \in F$, f has an approximate differential

$$(2.30) \quad apDf(x) : T_x F \rightarrow P^1$$

(c.f. [8], Thm 3.2.19). Moreover this differential is such that $\|\wedge_2 apDf(x)\| \leq 1$ almost everywhere, because f is 1-Lipschitz.

Now we can apply the area formula to f , (c.f. [8] Cor 3.2.20). For all $H^2|_F$ -integrable functions $g : F \rightarrow \mathbb{R}$, we have

$$(2.31) \quad \int_F (g \circ f) \cdot \|\wedge_2 apDf(x)\| dH^2 = \int_{P^1} g(z) N(f, z) dH^2 z,$$

where $N(f, z) = \# \{f^{-1}(z)\}$, and for $z \in p^1(F)$ we have $N(f, z) \geq 1$. Take $g \equiv 1$, we get

$$(2.32) \quad \int_F \|\wedge_2 apDf(x)\| dH^2 = \int_{P^1} N(f, z) dH^2 z \geq \int_{p^1(F)} dH^2 = H^2(p^1(F)).$$

Recall that p^1 is linear, hence its differential is itself. As a result $apDf(x)$ is the restriction of p^1 on the 2-subspace $T_x F$, which implies that if $\{u, v\}$ is an orthonormal basis of $T_x F$, then

$$(2.33) \quad \|\wedge_2 apDf(x)\| = \|\wedge_2 p^1(u \wedge v)\| = |p^1(T_x F)|$$

by (2.6). Hence by (2.32)

$$(2.34) \quad \int_F |p^1(T_x F)| dH^2(x) \geq H^2(p^1(F)).$$

A similar argument gives also:

$$(2.35) \quad \int_F |p^2(T_x F)| dH^2(x) \geq H^2(p^2(F)).$$

Summing over $i = 1, 2$ we get

$$(2.36) \quad \begin{aligned} H^2(p^1 F) + H^2(p^2 F) &\leq \int_F |p^1 T_x F| + |p^2 T_x F| dH^2(x) \\ &\leq \int_F \lambda dH^2(x) = \lambda H^2(F) \end{aligned}$$

since $|p^1 T_x F| + |p^2 T_x F| \leq \lambda$. □

Here are two corollaries of Lemma 2.27.

Corollary 2.37. *The set $P_0 = P_0^1 \cup_\perp P_0^2 \subset \mathbb{R}^4$ is minimal.*

Proof.

Let F be an Al competitor of P_0 in $B(0,1)$. By Remark 1.19, this means that there exists a Lipschitz deformation φ in \mathbb{R}^4 , with

$$(2.38) \quad \varphi|_{B(0,1)^c} = Id, \quad \varphi(B(0,1)) \subset B(0,1), \quad \text{and} \quad \varphi(P_0) \cap B(0,1) = F.$$

We will compare the measure of F with that of $P_0 \cap B(0,1)$.

Denote by $p_0^i, i = 1, 2$ the projections of \mathbb{R}^4 on P_0^i . Since F is 2-rectifiable, the approximate tangent plane $T_x F$ of F at x exists for almost all $x \in F$. By representing planes by unit simple 2-vectors, and by Lemma 2.14,

$$(2.39) \quad |p_0^1(T_x F)| + |p_0^2(T_x F)| \leq 1.$$

As a result, Lemma 2.27 gives

$$(2.40) \quad H^2(p_0^1(F)) + H^2(p_0^2(F)) \leq H^2(F).$$

Denote by $F^i = p_0^i \circ \varphi(P_0^i \cap B(0,1)), i = 1, 2$, then F^i is a deformation of $P_0^i \cap B(0,1)$, and by the minimality of planes (in arbitrary ambient dimension) we have

$$(2.41) \quad H^2(F^i) \geq H^2(P_0^i \cap B(0,1)).$$

Now since

$$(2.42) \quad p_0^i(F) = p_0^i \circ \varphi(P_0 \cap B(0,1)) \supset p_0^i \circ \varphi(P_0^i \cap B(0,1)) = F^i$$

we have

$$(2.43) \quad H^2(p_0^i(F)) \geq H^2(F^i) \geq H^2(P_0^i \cap B(0,1)).$$

We sum over i and get

$$(2.44) \quad \begin{aligned} H^2(F) &\geq H^2(p_0^1 F) + H^2(p_0^2 F) \\ &\geq H^2(P_0^1 \cap B(0,1)) + H^2(P_0^2 \cap B(0,1)) = H^2(P_0 \cap B(0,1)). \end{aligned}$$

This implies that P_0 is minimal. □

Corollary 2.45. *Suppose $\epsilon > 0$ is such that $\arccos(\epsilon/2) \leq \alpha_1 \leq \alpha_2$, and P^1, P^2 are two planes with characteristic angles (α_1, α_2) . Denote by p^i the orthogonal projection on $P^i, i = 1, 2$. Then if E is a closed 2-rectifiable set satisfying $p^i(E) \supset B(0,1) \cap P^i$, we have*

$$(2.46) \quad H^2(E) \geq \frac{2\pi}{1+\epsilon}.$$

Proof. by Proposition 2.19, we know that for almost all point $x \in E$, the approximate tangent plane of E at x $T_x E$ satisfies

$$(2.47) \quad |p^1(T_x E)| + |p^2(T_x E)| \leq 1 + \epsilon.$$

We apply Lemma 2.27 to E and obtain the conclusion. □

3 Uniqueness of P_0

In this section, we will add more information on Corollary 2.37. More precisely, we will prove the following Theorem 3.1. It looks quite natural, but its proof is more complicated than the author imagined.

Theorem 3.1. *[uniqueness of P_0] Let $P_0 = P_0^1 \cup_\perp P_0^2$, as in the previous section, and denote by p_0^i the orthogonal projection on P_0^i , $i = 1, 2$. Let $E \subset \overline{B}(0, 1)$ be a 2-dimensional closed reduced set such that $E \cap B(0, 1)$ is minimal in $B(0, 1) \subset \mathbb{R}^4$, and*

$$(3.2) \quad p_0^i(E \cap \overline{B}(0, 1)) \supset P_0^i \cap \overline{B}(0, 1);$$

$$(3.3) \quad E \cap \partial B(0, 1) = P_0 \cap \partial B(0, 1);$$

$$(3.4) \quad H^2(E \cap B(0, 1)) \leq 2\pi$$

(or equivalently $H^2(E \cap B(0, 1)) = 2\pi$, see near (3.5)).

Then $E = P_0 \cap \overline{B}(0, 1)$.

In the rest of the section, we suppose that E is a set that verifies all the hypotheses in Theorem 3.1.

First notice that $p_0^i(E) = P_0^i \cap \overline{B}(0, 1)$, because $E \subset \overline{B}(0, 1)$ and $p_0^i(E) \supset P_0^i \cap \overline{B}(0, 1)$. As a result

$$(3.5) \quad \begin{aligned} H^2(E) &= H^2(E \cap B(0, 1)) = 2\pi \\ &= H^2(P_0^1 \cap B(0, 1)) + H^2(P_0^2 \cap B(0, 1)) = H^2(p_0^1(E)) + H^2(p_0^2(E)). \end{aligned}$$

Compare with (2.36), and observe that here by Lemma 2.14 we can take $\lambda = 1$, we have that all the inequalities in (2.36), and hence also in (2.32), are in fact equalities. This means that

$$(3.6) \quad \text{for almost all point } x \in E, \quad |p_0^1(T_x E)| + |p_0^2(T_x E)| = 1, \text{ or equivalently } T_x E \in P(\Xi),$$

and

$$(3.7) \quad \text{for } i = 1, 2, \text{ for almost all } z \in p_0^i(E), \quad N(p_0^i, z) = \sharp\{p_0^{i-1}(z) \cap E\} = 1.$$

Now we are going to use these two conditions and Lemma 2.17 to get useful local properties of the set E .

First of all, since $E \cap B(0, 1)$ is a reduced minimal set in $B(0, 1)$, we know that for all $x \in E$, there exists a bi-Hölder ball $B(x, r) \subset B(0, 1)$, and in $B(x, r)$ the set E is bi-Hölder equivalent to a minimal cone C_x (c.f. Theorem 1.32). The cone C_x is a blow-up limit of E at x .

On the other hand, all that we know generally for a 2-dimensional minimal cone C is that its intersection with the unit sphere $S = C \cap \partial B(0, 1)$ is a finite collection of great circles and arcs of

great circles. Every great circle is disjoint from the rest of S ; at their ends, the arcs meet by set of three, with 120° angles, and in particular no free ends exists (c.f.[5], Proposition 14.1). Hence each endpoint of any of these arcs is a one-dimensional \mathbb{Y} point of the net S . Thus if a minimal cone is not the union of several transversal planes, its intersection with the unit sphere contains at least a \mathbb{Y} point. As a result, there exist \mathbb{Y} points in C arbitrarily near the origin, since C is a cone.

This discussion yields the following.

Lemma 3.8. *There is no point of type \mathbb{Y} in $E \cap B(0, 1)$.*

Proof. If $x \in E$ is a point of type \mathbb{Y} , then it means that the (unique, by Theorem 1.35 and Remark 1.36) tangent cone C_x is composed of 3 closed half planes $\{P_i\}_{1 \leq i \leq 3}$ which meet along a line D passing through the origin. Denote by $Q_i, 1 \leq i \leq 3$ the plane that contains P_i . In this case we claim that

$$(3.9) \quad \text{at least one } Q_i \text{ doesn't belong to } P(\Xi).$$

In fact, if we denote by v the unit vector generating D , then there exist three unit vectors $w_i, 1 \leq i \leq 3$ such that $w_i \perp v$, $Q_i = P(v \wedge w_i)$, and the angle between any two of the w_i is 120° .

If $Q_1 \notin P(\Xi)$, then the claim (3.9) holds automatically. Suppose that $Q_1 \in P(\Xi)$, and hence $v \wedge w_1 \in \Xi$. Then by Lemma 2.17, there exist unit vectors $v_j, u_j, j = 1, 2$ with $v_j \in P_0^1, u_j \in P_0^2$, $v_1 \perp v_2, u_1 \perp u_2$, and $\alpha \in [0, \frac{\pi}{2}]$, such that

$$(3.10) \quad v = \cos \alpha v_1 + \sin \alpha u_1, w_1 = \cos \alpha v_2 + \sin \alpha u_2.$$

But then since $u_j, v_j, j = 1, 2$ generate \mathbb{R}^4 , there exists $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 = 1$ such that $w_2 = av_1 + bv_2 + cu_1 + du_2$. Therefore

$$(3.11) \quad \begin{aligned} |p_0^1(Q_2)| + |p_0^2(Q_2)| &= |\wedge_2 p_0^1(v \wedge w_2)| + |\wedge_2 p_0^2(v \wedge w_2)| \\ &= |b \cos \alpha| + |d \sin \alpha| \leq (b^2 + d^2)^{\frac{1}{2}} (\cos^2 \alpha + \sin^2 \alpha)^{\frac{1}{2}} \\ &= (b^2 + d^2)^{\frac{1}{2}} \leq 1. \end{aligned}$$

So if we want Q_2 belong to $P(\Xi)$, all inequalities in (3.11) should be equalities, therefore $b^2 + d^2 = 1$, and $b : d = \pm \cos \alpha : \sin \alpha$. As a result, $(b, d) = (\pm \cos \alpha, \pm \sin \alpha)$, $a^2 + c^2 = 0$, and hence $w_2 = \pm w_1$ or $\pm(\cos \alpha v_2 - \sin \alpha u_2)$.

Notice that angle between w_1 and w_2 is 120° , so there are only two possibilities for α , which are $\frac{\pi}{3}$ and $\frac{\pi}{6}$.

If $\alpha = \frac{\pi}{3}$, then $w_1 = \frac{1}{2}v_2 + \frac{\sqrt{3}}{2}u_2$ and $w_2 = \frac{1}{2}v_2 - \frac{\sqrt{3}}{2}u_2$. But the argument above holds also for w_3 , so we have $w_3 = w_2$, which contradicts the fact that the angle between w_2 and w_3 is 120° .

The situation is the same for $\alpha = \frac{\pi}{6}$.

Thus we have proved the claim (3.9).

Now suppose for example that $Q_1 \notin P(\Xi)$. Then since $P(\Xi)$ is closed in $G(4, 2)$, there exists an open set $U \subset G(4, 2)$ that contains Q_1 such that $U \cap P(\Xi) = \emptyset$.

As we said, \mathbb{Y} is a cone with the full-length property. Hence by the theorem of $C^{1+\alpha}$ -regularity for minimal sets (c.f. Theorem 1.35), there exists $r > 0$ such that $B(x, r) \subset B(0, 1)$, and in $B(x, r)$, E coincides with the image of $C_x + x$ of a C^1 homeomorphism φ from \mathbb{R}^4 to \mathbb{R}^4 (recall that $C_x = \cup_{i=1}^3 P_i$). Denote by $R = \varphi(Q_1)$. Then the map $T : R \rightarrow G(4, 2)$, $T(x) = T_x R$ is continuous. As a result $T^{-1}(U)$ is open. Denote $R_+ = \varphi(P_1)$. Then $T^{-1}(U) \cap \varphi^{-1}(R_+) \neq \emptyset$, since it contains x . Therefore $T^{-1}(U)$ is relatively open in R_+ . But R_+ is a C^1 manifold with boundary, whose boundary $\varphi(D)$ is of zero measure, so $R_+ \cap T^{-1}(U) \setminus \varphi(D)$ is of positive measure. Now for every $y \in R_+ \cap T^{-1}(U) \setminus \varphi(D)$, $T_y R = T_y E \notin P(\Xi)$, thus we have found a set of positive measure and all of its point do not satisfy (3.6); this gives a contradiction. \square

After this lemma, we will obtain some useful description of the local structure around each point x of E .

Lemma 3.12. *For each $x \in E \cap B(0, 1)$, every blow-up limit of E at x is either a plane belonging to $P(\Xi)$, or P_0 .*

Proof. Suppose that X is a blow-up limit of E at x , suppose also that $x = 0$ for short. First we claim that

$$(3.13) \quad X \text{ doesn't contain any point of type } \mathbb{Y}.$$

Suppose this is not true, then there exists $p \in X$ such that p is of type \mathbb{Y} . Then p is not the origin, because otherwise X is of type \mathbb{Y} , and hence 0 is of type \mathbb{Y} , which gives a contradiction with Lemma 3.8.

So p is not the origin. Then since X is a cone, for every $r > 0$, $rp \in X$ is a point of type \mathbb{Y} . We can thus suppose that $\|p\| = 1$. Then by our description of 2-dimensional minimal cones (above Lemma 3.8), there exists $0 < r < \frac{1}{2}$ such that in $B(p, r)$, X coincides with a cone Y of type \mathbb{Y} centered at p .

Define $d_{x,r}(E, F) = \frac{1}{r} \max\{\sup\{d(y, F) : y \in E \cap B(x, r)\}, \sup\{d(y, E) : y \in F \cap B(x, r)\}\}$, which is the relative distance of two sets E, F with respect to the ball $B(x, r)$. Now X is a blow-up limit of E at 0 , so that there exists $s > 0$ (large) such that $d_{0,2}(X, sE) < \frac{r\epsilon_2}{100}$, where ϵ_2 is the constant in Proposition 16.24 of [5]. Equivalently, $d_{p,\frac{r}{2}}(sE, X) < \frac{\epsilon_2}{50}$.

We want to show that

$$(3.14) \quad d_{p,\frac{r}{2}}(sE, X) = d_{p,\frac{r}{2}}(sE, Y).$$

Once we have this, we take a point $z \in sE$ such that $d(z, p) < \frac{r}{2} \times \frac{\epsilon_2}{50}$, then $d_{z,\frac{r}{4}}(sE, Y + z - p) < \frac{\epsilon_2}{10}$. Here $Y + z - p$ is a \mathbb{Y} cone centered at z . But sE is minimal (since E is), therefore Proposition 16.24

of [5] gives that sE contains a \mathbb{Y} point, and hence E , too. This contradicts Lemma 3.8. Thus we obtain our claim (3.13).

We should still prove (3.14). By definition,

$$(3.15) \quad d_{p, \frac{r}{2}}(sE, X) = \frac{2}{r} \max\left\{ \sup_{x \in sE \cap B(p, \frac{r}{2})} d(x, X), \sup_{x \in X \cap B(p, \frac{r}{2})} d(x, sE) \right\}.$$

For the second term, we have $X \cap B(p, \frac{r}{2}) = Y \cap B(p, \frac{r}{2})$, and hence

$$(3.16) \quad \sup_{x \in X \cap B(p, \frac{r}{2})} d(x, sE) = \sup_{x \in Y \cap B(p, \frac{r}{2})} d(x, sE).$$

For the first term, we have

$$(3.17) \quad d(x, X) = d(x, X \cap B(p, r)) \text{ for all } x \in sE \cap B(p, \frac{r}{2}).$$

In fact, since $d_{p, \frac{r}{2}}(sE, X) < \frac{\epsilon_2}{50}$, for each $x \in sE \cap B(p, \frac{r}{2})$, $d(x, X) < \frac{\epsilon_2}{50} \times \frac{2}{r}$, so $d(x, X) = d(x, X \cap B(x, \frac{\epsilon_2}{50} \times \frac{2}{r})) \leq d(x, X \cap B(p, r))$, because $B(x, \frac{\epsilon_2}{50} \times \frac{2}{r}) \subset B(p, r)$. On the other hand, $X \cap B(p, r) \subset X$, therefore $d(x, X) \geq d(x, X \cap B(p, r))$. Thus we have (3.17), and hence

$$(3.18) \quad \begin{aligned} \sup_{x \in sE \cap B(p, \frac{r}{2})} d(x, X) &= \sup_{x \in sE \cap B(p, \frac{r}{2})} d(x, X \cap B(p, r)) \\ &= \sup_{x \in sE \cap B(p, \frac{r}{2})} d(x, Y \cap B(p, r)) \geq \sup_{x \in sE \cap B(p, \frac{r}{2})} d(x, Y). \end{aligned}$$

Combining with (3.16) we have

$$(3.19) \quad d_{p, \frac{r}{2}}(sE, X) \leq d_{p, \frac{r}{2}}(sE, Y).$$

A similar argument yields

$$(3.20) \quad d_{p, \frac{r}{2}}(sE, Y) \leq d_{p, \frac{r}{2}}(sE, X).$$

So we have (3.14), and (3.13) follows.

Since X is a minimal cone, as we have said before, $X \cap \partial B(0, 1)$ is a finite collection of great circles and arcs of great circles that meet by 3 with angles of 120° . Then (3.13) implies that there is no such arcs, since X does not have \mathbb{Y} points. As a result, $X \cap \partial B(0, 1)$ is a finite collection of great circles, and therefore X is the union of a finite number of transversal planes.

By Remark 1.36, X is a full-length cone, so the C^1 regularity holds (Thm 1.35). Then by the same argument as for \mathbb{Y} in the proof of Lemma 3.8, every plane in X belongs to $P(\Xi)$, since $P(\Xi)$ is closed.

As a result, if X is not a plane, then $X = \cup_{i=1}^n Q_n$ with $Q_n \in \Xi$ with $n \geq 2$. These Q_n are transversal, by Lemma 2.17. Moreover in a small neighborhood $B(x, r)$ of the point x , the set E is a union of n transversal C^1 manifolds $S_i, 1 \leq i \leq n$, and the tangent plane to S_i at x is Q_n .

By Lemma 2.17, for all $Q \in \Xi$ and $Q \neq P_0^2$, there exists $s = s_Q > 0$ such that $p_0^1(Q \cap B(0, 1)) \supset P_0^1 \cap B(0, s_Q)$. Hence if X contains two planes $Q_i, Q_j, i \neq j$, that are different from P_0^2 , there exists $s > 0$ such that $p_0^1(Q_1 \cap B(0, 1)) \cap p_0^1(Q_2 \cap B(0, 1)) \supset B(0, s) \cap P_0^1$. Then since the tangent plane to S_i at x is Q_i and that to S_j at x is Q_j , there exists a neighborhood U of $p_0^1(x)$ which is contained in both projections $p_0^1(S_i)$ and $p_0^1(S_j)$. Notice that S_i and S_j are transverse, hence $S_i \cap S_j = \{x\}$. This yields that each point $y \in U \setminus p_0^1(x)$ admits at least two pre-images by p_0^1 in $E \cap B(x, r)$, one is in S_i and the other is in S_j , i.e.,

$$(3.21) \quad (U \setminus \{p_0^1(x)\}) \cap P_0^1 \subset \{z \in P_0^1, \# \{p_0^{1^{-1}}\{z\} \cap E\} \geq 2\}.$$

But U is open, so $(U \setminus \{p_0^1(x)\}) \cap P_0^1$ is of positive measure, which contradicts (3.7).

Hence in X , we have at most one plane different from P_0^2 . The same argument gives also that we have at most one plane which is not P_0^1 . But X contains at least two planes, therefore $X = P_0^1 \cup P_0^2 = P_0$. \square

Lemma 3.12 says that there exists only two types of blow-up limits in E , and both of them have the full-length property. As a result, by Thm 1.35, around each point $x \in E$, E is locally C^1 equivalent to a plane or to P_0 .

The two next lemmas will describe more precisely what happens around each of the two types of singularities. We identify \mathbb{R}^4 with $\mathbb{C} \times \mathbb{C}$.

Lemma 3.22. *Let $\{e_1, e_2 = ie_1, e_3, e_4 = ie_3\}$ be an orthonormal basis of \mathbb{R}^4 , with $P_0^1 = e_1 \wedge e_2$ and $P_0^2 = e_3 \wedge e_4$. Let $x \in E \cap B(0, 1)$ be a point of type \mathbb{P} such that $T_x E \neq P_0^2$. Then there exists $r = r(x) > 0$ such that in $B(x, r)$, E is (under the given basis) the graph of a complex analytic or anti-analytic function $\varphi = \varphi_x : P_0^1 \rightarrow P_0^2$. More precisely,*

$$(3.23) \quad E \cap B(x, r) = \text{graph}(\varphi) \cap B(x, r).$$

We have also a similar conclusion for x such that $T_x E \neq P_0^1$, i.e., near x , E is the graph of a complex analytic or anti-analytic function from P_0^2 to P_0^1 .

Proof. We will only prove it for $T_x E \neq P_0^2$. The other case is similar.

So let $x \in E$ be as in the lemma. Assume all the hypotheses in the lemma. Since x is a \mathbb{P} point, there exists $r_1 > 0$ such that in $B(x, r_1)$, E is the graph of a C^1 function $\varphi_1 : T_x E \rightarrow T_x E^\perp$ (c.f. Thm 1.35). If we denote by π the projection from \mathbb{R}^4 to $T_x E$, and define $F : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ by $F(y) = y - \varphi_1(\pi(y)) \in T_x E^\perp \cong \mathbb{R}^2$ for $y \in B(x, r_1)$, then F is C^1 regular, and $E \cap B(x, r_1) = \{y \in B(x, r_1) : F(y) = (0, 0)\}$. As a result, $DF(x)$ is a linear map from \mathbb{R}^4 to \mathbb{R}^2 , with $T_x E = \text{Ker } DF(x)$.

By the hypothesis, $T_x E \neq P_0^2$, hence by Lemmas 3.12 and 2.17, $T_x E$ can be represented by $T_x E = P((\cos \alpha v_1 + \sin \alpha u_1) \wedge (\cos \alpha v_2 + \sin \alpha u_2))$, with $\alpha \in [0, \frac{\pi}{2})$, $v_i, u_i, i = 1, 2$ unit vectors,

$v_i \in P_0^1, u_i \in P_0^2$ and $v_1 \perp v_2, u_1 \perp u_2$. In particular, $P_0^2 \cap \text{Ker} DF(x) = P_0^2 \cap T_x E = \{0\}$. This means that $DF(x)|_{P_0^2}$ is invertible. Then by the implicit function theorem, there exists $r_2 > 0$ and a C^1 map $\varphi : P_0^1 \rightarrow P_0^2$, such that for every $y = (y_1, y_2) \in P_0^1 \times P_0^2 \cap B(x, r_2)$, $F(y) = 0 \Leftrightarrow y_2 = \varphi(y_1)$. As a result, in $B(x, r_2)$, E is the graph of $\varphi : P_0^1 \rightarrow P_0^2$.

We still have to prove that φ is complex analytic or anti-analytic. This will follow from a particular property of $P(\Xi)$.

More precisely, return to the orthonormal basis $\{e_1, e_2 = ie_1, e_3, e_4 = ie_3\}$ with $P_0^1 = e_1 \wedge e_2$ and $P_0^2 = e_3 \wedge e_4$. Use Lemma 2.17, and expand each unit vector under the basis; a simple calculation gives

$$(3.24) \quad \begin{aligned} \Xi &= \{\pm[ae_1 + be_2 + ce_3 + de_4] \wedge [-be_1 + ae_2 \pm (-de_3 + ce_4)], \\ &\quad a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 = 1\}, \end{aligned}$$

and hence

$$(3.25) \quad \begin{aligned} P(\Xi) &= \{P([ae_1 + be_2 + ce_3 + de_4] \wedge [-be_1 + ae_2 \pm (-de_3 + ce_4)]), \\ &\quad a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 = 1\}. \end{aligned}$$

If $T_y E = [ae_1 + be_2 + ce_3 + de_4] \wedge [-be_1 + ae_2 + (-de_3 + ce_4)]$, this means that $d\varphi(y)(a + bi) = c + di, d\varphi(y)(-b + ai) = -d + ci$. But $d\varphi(y)$ is (real) linear, and $a^2 + b^2 \neq 0$, therefore $a + bi, -b + ai$ is a basis, and $d\varphi(y)$ is determined by its values at these two points. Notice that $\frac{c+di}{a+bi} = \frac{-d+ci}{-b+ai}$, and denote it by $A \in \mathbb{C}$, then we have $d\varphi(y)(z) = Az$, which is complex analytic. In other words,

$$(3.26) \quad \frac{d\varphi}{d\bar{z}}(y) = 0.$$

If $T_y E = [ae_1 + be_2 + ce_3 + de_4] \wedge [-be_1 + ae_2 - (-de_3 + ce_4)]$, a similar argument gives $d\varphi(y)(z) = B\bar{z}$, and hence

$$(3.27) \quad \frac{d\varphi}{dz}(y) = 0.$$

So we have proved that φ is a complex C^1 function such that at each point, it is either analytic, or anti-analytic.

Denote by $B = B(p^1(x), r) \cap P_0^1$, $B_1 = \{y \in B, \frac{d\varphi}{dz}(y) \neq 0\}$. Then B_1 is open, since φ is C^1 . If $B_1 = \emptyset$, then φ is anti-analytic. Otherwise, B_1 is not empty, and set $g = \frac{\partial \varphi}{\partial z}$. Then g is continuous on B , and $B_1 = \{y \in B : g(y) \neq 0\}$. Moreover, since $\frac{d\varphi}{dz}(y) \neq 0$ on B_1 , $\frac{d\varphi}{d\bar{z}}(y) = 0$ on B_1 (since at each point φ is either analytic or anti-analytic), and hence φ is holomorphic on the open set B_1 , so that its derivative g is holomorphic, too. Then conclusion of Lemma 3.22 will follow from the following theorem (c.f.[19] Thm 12.14) :

Theorem 3.28 (Radó's theorem). *Let $U \subset \mathbb{C}$ be an open domain, and f be a continuous function on \bar{U} . Set $\Omega = \{z \in U : f(z) \neq 0\}$, and suppose that f is holomorphic on Ω . Then f is holomorphic on U .*

In fact, we apply the theorem to g , and obtain that g is holomorphic on B . But since $B_1 \neq \emptyset$, $g \not\equiv 0$. As a result $B_1^C = \{y \in B : g(y) = 0\}$ doesn't have any limit point. Notice that $B_1^C \supset B_2 := \{y \in B, \frac{d\varphi}{dz}(y) \neq 0\}$, and B_2 is open, so it is an open set without limit point. Therefore $B_2 = \emptyset$, which means that φ is complex analytic on B .

Hence φ is complex analytic or anti-analytic on $B = B(p^1(x), r) \cap P_0^1$. \square

Lemma 3.29. *If $x \in E \cap B(0, 1)$ is of type P_0 , then there exists $r = r(x) > 0$ such that*

$$(3.30) \quad E \cap B(x, r) = (P_0 + x) \cap B(x, r).$$

Proof. Let $B(x, r')$ be a C^1 ball for the point x , in which E coincides with the image of $P_0 + x$ by a C^1 function $\varphi : E \cap B(x, r') = \varphi((P_0 + x) \cap B(x, r'))$, where φ is a diffeomorphism and $\varphi(x) = x$, $D\varphi(x) = Id$ (c.f. Thm 1.35). Set $A_i = \varphi(P_0^i + x) \cap B(x, r')$, $i = 1, 2$. Then A_i are transversal C^1 manifolds, $[A_1 \cup A_2] \cap B(x, r') = E \cap B(x, r')$ and $A_1 \cap A_2 = \{x\}$. Moreover, the tangent plane to A_i at the point x is P_0^i , $i = 1, 2$. In particular, $P_0^1(A_1)$ contains a neighborhood U of $p_0^1(x)$ in P_0^1 .

By a similar argument as the one in Lemma 3.22, there exists $r_2 < \frac{r_1}{2}$ such that in $B(x, r_2)$, A_2 is the graph of the analytic or anti-analytic complex function $\psi : P_0^2 + x \rightarrow P_0^1$, where $\psi = p_0^1 \circ \varphi$ on $P_0^2 + x \cap B(x, r_2)$. Without loss of generality, suppose that it is complex analytic.

If ψ is not constant, it will be an open map, because it is analytic. Therefore $p_0^1(A_2)$ also contains a neighborhood U' of $p_0^1(x)$ in P_0^1 . But $A_1 \cap A_2 = \{x\}$, hence for every point $y \in U \cap U' \setminus \{p_0^1(x)\}$, $p_0^{1^{-1}}\{y\}$ admits at least two points in E , one in A_1 , and the other in A_2 . But $U \cap U' \setminus \{p_0^1(x)\}$ is of non zero measure, this contradicts (3.7).

Hence ψ is constant. As a result,

$$(3.31) \quad \varphi((P_0^2 + x) \cap B(x, r_2)) = P_0^2 \cap B(x, r_2).$$

We can do the same for $\varphi(P_0^1 + x)$, and we obtain that there exists $r < r_2$ such that in $B(x, r)$, $\varphi(P_0^1 + x)$ is $P_0^1 + x$ itself. This completes the proof of Lemma 3.29. \square

Let us sum up a little. The minimal set E has only two types of points : planar points, around which the set E is a C^1 (locally analytic or anti analytic) manifold; and P_0 points, around which E is locally P_0 itself. Hence in fact,

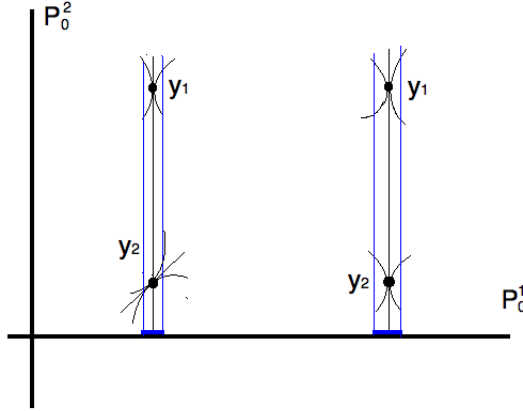
$$(3.32) \quad \begin{aligned} &\text{The set } E \text{ is composed of countably many transversal } C^1 \text{ manifolds } S_1, S_2, \dots, S_n, \dots. \\ &\text{They are locally analytic or anti analytic, and at any of their intersections,} \\ &E \text{ is locally a translation of } P_0. \end{aligned}$$

We know that (3.7) holds for almost all $z \in p_0^1(E)$. But now let us look at those points x in E where the projection p_0^1 is not injective.

Lemma 3.33. *Suppose that $y_1, y_2 \in E \cap B(0, 1)$ are such that $p_0^1(y_1) = p_0^1(y_2)$. Then at least one of the two y_i is such that E coincides with $P_0^2 + y_i$ in a neighborhood of y_i .*

Proof. By the argument near (3.21), at least one of the tangent planes $T_{y_1}E$, $T_{y_2}E$ is P_0^2 . Suppose $T_{y_1}E = P_0^2$ for example. Then by Lemma 3.22, there exists $s_1 > 0$ such that in $B(y_1, s_1)$, E is the graph of a complex analytic or anti-analytic function (under the basis given before) $\varphi_1 : P_0^2 \rightarrow P_0^1$, with $\varphi_1(0) = p_0^1(y_1)$, $D\varphi_1 = 0$.

First, if the tangent plane $T_{y_2}E$ of E at y_2 is not P_0^2 , then again by the argument near (3.21), there exists $s_2 > 0$ such that $B(y_1, s_1) \cap B(y_2, s_2) = \emptyset$, and $B(p_0^1(y_2), s_1) \cap P_0^1 \subset p_0^1(B(y_2, s_2) \cap E)$. Therefore $p_0^1(B(y_1, s_1) \cap E)$ is of measure zero, because of (3.7) and $p_0^1(y_1) = p_0^1(y_2)$. Hence φ_1 has to be constant, otherwise it will be an open map, which means $p_0^1(B(y_1, s_1) \cap E)$ contains an open set in P_0^1 , which is of positive measure, and thus leads to a contradiction. So in $B(y_1, s_1)$, E coincides with $P_0^2 + y_1$. (See the picture 3-1 below, the two points on the left.)



3-1

If $T_{y_2}E$ is also P_0^2 , then in $B(y_2, s_2)$, E is the graph of a complex analytic or anti-analytic function $\varphi_2 : P_0^2 \rightarrow P_0^1$. Then at least one of the two $\varphi_i, i = 1, 2$ is constant, otherwise all of them are open maps, so that $p_0^1(B(y_1, s_1) \cap E) \cap p_0^1(B(y_2, s_2) \cap E)$ contains an open set in P_0^1 , which contradicts (3.7). \square

Lemma 3.34. *If there exists a point in $P_0^1 \cap B(0, 1)$ whose pre-image by p_0^1 contains more than one point in $E \cap B(0, 1)$, then $P_0^2 \cap B(0, 1) \subset E$.*

Proof.

Lemma 3.33 says that if there exists a point in $P_0^1 \cap B(0, 1)$ whose pre-image by p_0^1 contains more than one point in E , then there exists a piece of P_0^2 above it. Denote this piece by $(P_0^2 + x) \cap B(x, r)$. On the other hand, by (3.32), $E \cap B(0, 1)$ is composed of at most countably many transversal C^1 manifolds S_1, \dots, S_l, \dots , so there exists i such that $(P_0^2 + x) \cap B(x, r) \subset S_i$.

We claim that $S_i = (P_0^2 + x) \cap B(0, 1)$. So set $A = \{y \in S_i : \text{there exists } r = r_y > 0 \text{ such that } S_i \cap B(y, r) = (P_0^2 + y) \cap B(y, r)\}$. By definition, A is open in S_i . On the other hand, for any $y \in \bar{A} \cap S_i$, there exists a sequence $y_n \in A$ that converges to y . But S_i is a C^1 manifold, hence the tangent plane $T_y S_i = \lim_{n \rightarrow \infty} T_{y_n} S_i = P_0^2$. Therefore there exists $r > 0$ such that in $B(y, r)$, S_i is the graph of a complex analytic (the anti-analytic case is exactly the same) $\psi : P_0^2 \rightarrow P_0^1$. But $\psi' = 0$ on $\{p_0^2(y_n)\} \cup \{p_0^2(y)\}$, which, if $y \notin A$, is an infinite set with a limit point. So $\psi' \equiv 0$ in $p_0^2(B(y, r))$, so that ψ is constant, which yields that $y \in A$. So A is closed. But S_i is connected, A is open and closed and non-empty, hence $A = S_i$. But S_i is a C^1 manifold, so the only possibility is that it is a piece of $(P_0^2 + x) \cap B(0, 1)$. However, S_i is both closed and open in $(P_0^2 + x) \cap B(0, 1)$, because A is both open and closed in $(P_0^2 + x) \cap B(0, 1)$ by the same argument as above, and $S_i = A$. As a result, $S_i = (P_0^2 + x) \cap B(0, 1)$.

By (3.3), $\bar{S}_i \cap \partial B(0, 1) \subset E \cap \partial B(0, 1) \subset P_0 \cap \partial B(0, 1)$, since E is closed. This implies that $S_i = P_0^2 \cap B(0, 1)$. Hence $P_0^2 \cap B(0, 1) \subset E$. \square

Remark 3.35. By a similar discussion, if p_0^2 is not injective on $E \cap B(0, 1)$, then $P_0^1 \cap B(0, 1) \subset E$.

Lemma 3.36. The set E contains at least one point of type P_0 .

Proof. First of all, by Lemma 3.34 and Remark 3.35, if neither of the $p_0^i, i = 1, 2$ is injective, then $P_0 \cap B(0, 1) = (P_0^1 \cup P_0^2) \cap B(0, 1) \subset E$. In particular, there exists a P_0 point.

Thus if E contains no P_0 point, then at least one of the $p_0^i, i = 1, 2$ is injective. Without loss of generality, we suppose p_0^1 is injective on $E \cap B(0, 1)$, and E contains no P_0 point. Notice that (3.2) is true for $E \cap \bar{B}(0, 1)$, and we know that $p_0^1(E \cap \partial B(0, 1)) = \{0\} \cup (P_0^1 \cap \partial B(0, 1))$, hence $p_0^1(E \cap B(0, 1)) \supset P_0^1 \cap B(0, 1) \setminus \{0\}$. Hence the injectivity gives

$$(3.37) \quad \text{for all } z \in P_0^1 \cap B(0, 1) \setminus \{0\}, \# \{p_0^{1^{-1}}(z) \cap E \cap B(0, 1)\} = 1,$$

and

$$(3.38) \quad \# \{p_0^{1^{-1}}(0) \cap E \cap B(0, 1)\} \leq 1.$$

So there exists $\psi : P_0^1 \cap B(0, 1) \setminus \{0\} \rightarrow P_0^2$ such that $E \cap B(0, 1)$ is its graph. Therefore above those points $y \in E$ whose tangent planes $T_y E \neq P_0^2$, ψ is locally a complex analytic or anti-analytic function (by Lemma 3.22).

We claim that there is no singular point $y \in E \cap B(0, 1)$ with $T_y E = P_0^2$. In fact, if y is a such point, then by Lemma 3.22 (recall that we have supposed that E contains no P_0 point), in a neighborhood $B(y, r)$, E is the graph of a complex analytic or anti-analytic function g from P_0^2 to P_0^1 . Suppose $p_0^2(y) = 0$ for simplicity, hence $g'(0) = 0$. Then 0 is a zero of $g - g(0)$ of order at least 2. Thus in a punctured neighborhood O of $g(0) \in \mathbb{C}$, each point has at least 2 pre-images. So

$O \subset \{z \in P_0^1 : \#\{p_0^{1^{-1}}\{z\} \cap E\} \geq 2\}$. But O is of positive measure because it is open. This contradicts (3.7).

So there is no singular point for ψ , therefore ψ is a C^1 function on $P_0^1 \cap B(0, 1) \setminus \{0\}$, and $E \cap B(0, 1)$ is its graph. For each point $x \in P_0^1 \cap B(0, 1) \setminus \{0\}$, ψ is analytic or anti analytic in a small neighborhood of x . Moreover ψ is bounded. By the same argument in Lemma 3.22, we know that ψ is globally analytic or anti-analytic on any compact subdomain of $P_0^1 \cap B(0, 1) \setminus \{0\}$, hence is globally analytic or anti-analytic on $P_0^1 \cap B(0, 1) \setminus \{0\}$. Suppose it is analytic. Then since ψ is bounded around $\{0\}$, we can extend ψ to $P_0^1 \cap B(0, 1)$. But since ψ is analytic with $\psi|_{\partial B(0, 1)} = 0$, the graph of ψ has to be $P_0^1 \cap B(0, 1)$, hence

$$(3.39) \quad E \cap B(0, 1) \subset \text{graph}(\psi) = P_0^1 \cap B(0, 1).$$

This contradicts the hypothesis (3.3).

Thus we obtain the existence of a point of type P_0 in E . □

Proof of Theorem 3.1. We know by Lemma 3.36 that E contains a point x of type P_0 . By Lemma 3.29, there exists $r > 0$ such that $E \cap B(x, r) = (P_0 + x) \cap B(x, r)$. But in this case, neither p_0^1 nor p_0^2 is injective on $E \cap B(0, 1)$. Hence by Lemma 3.34 and Remark 3.35, $P_0 \cap B(0, 1) = (P_0^1 \cup P_0^2) \cap B(0, 1) \subset E$. But $H^2(E) = H^2(P_0 \cap B(0, 1))$, and $E \cap B(0, 1)$ is reduced. As a result, $E \cap B(0, 1) = P_0 \cap B(0, 1)$. This completes the proof of Theorem 3.1. □

4 Existence of minimal sets

Now we begin to prove Theorem 1.2.

Suppose that the conclusion of Theorem 1.2 is not true. Then there exists a sequence of unions of 2 planes $P_k = P_k^1 \cup_{\theta_k} P_k^2 \subset \mathbb{R}^4$ which are not minimal, with $\theta_k \geq \frac{\pi}{2} - \frac{1}{k}$. Moreover, we can suppose that all the P_k^1 are equal to P_0^1 .

Our proof would be simpler if we could find a sequence of minimizers E_k such that for each k , E_k minimizes the measure among all deformations of $P_k \cap \overline{B}(0, 1)$ in $B(0, 1)$. (We shall call such a set a solution of Plateau's problem). It is a standard fact about the Hausdorff distance that we can find a subsequence (which we shall still denote by $\{E_k\}$ to save notation) such that $\{E_k\}$ converges to a limit E_∞ . Since $P_k \cap \partial B(0, 1) = E_k \cap \partial B(0, 1)$, and $P_k \cap \partial B(0, 1) \rightarrow P_0 \cap \partial B(0, 1)$, we can hope to prove that the boundary of E_∞ is $P_0 \cap \partial B(0, 1)$. Moreover, since each E_k is minimal in $B(0, 1)$, their limit E_∞ has to be minimal in $B(0, 1)$, too (c.f.[4] Thm 4.1). Notice also that (3.2) is true since E_∞ is the limit of a sequence of deformations, we would deduce that $E_\infty = P_0 \cap \overline{B}(0, 1)$, by the uniqueness theorem 3.1. We could begin our stopping time argument as soon as we got such a sequence.

Unfortunately we do not know any theorem which guarantees the existence of a solution of Plateau's problem. So we have to work more to get through this. We are disposed of a weaker existence theorem:

Theorem 4.1 (Existence of minimal sets; c.f. [9], Thm 6.1.7). *Let $U \subset \mathbb{R}^n$ be an open domain, $0 < d < n$, and let \mathfrak{F} be a class of non-empty sets relatively closed in U and satisfying (1.17), which is stable by deformations in U . Suppose that*

$$(4.2) \quad \inf_{F \in \mathfrak{F}} H^d(F) < \infty.$$

Then there exists $M > 0$ (depends only on d and n), a sequence (F_k) of elements of \mathfrak{F} , and a set E of dimension d relatively closed in U that verifies (1.17), such that:

(1) There exists a sequence of compact sets $\{K_m\}_{m \in \mathbb{N}}$ in U with $K_m \subset K_{m+1}$ for all m and $\cup_{m \in \mathbb{N}} K_m = U$, such that

$$(4.3) \quad \lim_{k \rightarrow \infty} d_H(F_k \cap K_m, E \cap K_m) = 0 \text{ for all } m \in \mathbb{N};$$

(2) For all open sets V such that \bar{V} is relatively compact in U , from a certain rank,

$$(4.4) \quad F_k \text{ is } (M, +\infty)\text{-quasiminimal in } V;$$

(see Definition 4.5 below)

(3) $H^d(E) \leq \inf_{F \in \mathfrak{F}} H^d(F)$;

(4) E is minimal in U .

Definition 4.5 (Quasiminimal set). *Let $0 < d < n$ be two integers, $M > 0, \delta > 0$, let U be open in \mathbb{R}^n . The set E is said to be (M, δ) -quasi minimal in U ($E \subset QM(U, M, \delta)$ for short) if E is closed in U , (1.17) is true, and for every deformation φ in U as in Definition 1.11 such that $\text{diam}(\widehat{W}) < \delta$,*

$$(4.6) \quad H^d(E \cap W_1) \leq M H^d(\varphi_1(E \cap W_1)).$$

Remark 4.7. *In Theorem 4.1, we can also ask that $\{F_k\}$ is a minimizing sequence, i.e., $\lim_{k \rightarrow \infty} H^2(F_k) = \inf_{F \in \mathfrak{F}} H^d(F)$.*

Theorem 4.1 is a weaker result, which gives a certain type of minimizer, without preserving the boundary condition, and this minimizer may not be in the class \mathfrak{F} , either. But for our case, we do not need all these strong properties. Some weaker properties are sufficient for us to carry on the proof (as we will see soon).

Recall that $P_k = P_k^1 \cup_{\theta_k} P_k^2$ is a sequence of unions of two planes which are not minimal, with $\theta_k > \frac{\pi}{2} - \frac{1}{k}$. Moreover we suppose that all the P_k 's are the same. Thus $P_k^1 = P_0^1$. Denote by P_0^2 the plane orthogonal to P_0^1 , $P_0 = P_0^1 \cup_{\perp} P_0^2$. Then $P_k \cap \bar{B}(0, 1)$ converges to $P_0 \cap \bar{B}(0, 1)$ for the Hausdorff distance.

Proposition 4.8. *For each k , there exists a closed set $E_k \subset \overline{B}(0, 1)$ such that*

- (1) E_k is minimal in $\mathbb{R}^4 \setminus [P_k \setminus B(0, 1)]$;
- (2) $\partial B(0, 1) \cap E_k = \partial B(0, 1) \cap P_k$;
- (3) $p_k^i(E_k) \supset P_k^i \cap \overline{B}(0, 1)$, where p_k^i denotes the projection on $P_k^i, i = 1, 2$;
- (4) $H^2(E_k) < H^2(P_k \cap B(0, 1)) = 2\pi$;
- (5) E_k is contained in the convex hull of $P_k \cap \overline{B}(0, 1)$.

Proof of (1) and (4) of Proposition 4.8. Fix a k . Take $U = \mathbb{R}^4 \setminus [P_k \setminus B(0, 1)]$, and let \mathfrak{F} be the class of all deformations of $P_k \cap \overline{B}(0, 1)$ in U . Then by Theorem 4.1, and Remark 4.7, for $d = 2$, there exists a minimizing sequence of sets $F_l \in \mathfrak{F}$ that are also uniformly quasiminimal in U , with a uniform constant M . Moreover the sequence converges under the Hausdorff distance. Denote by E_k its limit. Then by the conclusion of Theorem 4.1, the terms (1) and (4) of Proposition 4.8 are automatically true. In (4) we have a strict inequality because we have supposed that P_k is not minimal. \square

For (3), we begin by proving the following lemma.

Lemma 4.9. *Let P be a plane in \mathbb{R}^4 , and p the projection on P . let φ be a Lipschitz mapping from \mathbb{R}^4 to \mathbb{R}^4 , such that $\varphi|_{P \cap B(0, 1)^c} = id$. Then*

$$(4.10) \quad p[\varphi(P \cap \overline{B}(0, 1))] \supset P \cap \overline{B}(0, 1).$$

Proof. We prove it by contradiction.

Suppose that there exists $x \in P \cap \overline{B}(0, 1)$ such that $x \notin p[\varphi(P \cap \overline{B}(0, 1))]$. Then $x \in P \cap B(0, 1)$, since $P \cap \partial B(0, 1) = \varphi(P \cap \partial B(0, 1)) \subset p[\varphi(P \cap \overline{B}(0, 1))]$ by hypothesis. Then there exists $r > 0$ such that $B(x, r) \subset B(0, 1)$. On the other hand, $P \cap \overline{B}(0, 1)$ is compact, hence its image $p[\varphi(P \cap \overline{B}(0, 1))]$ by the continuous map $p \circ \varphi$ is also compact, such that $\{p[\varphi(P \cap \overline{B}(0, 1))]\}^C$ is open. As a result, there exists $r' < r$ such that $B(x, r') \cap p[\varphi(P \cap \overline{B}(0, 1))] = \emptyset$. In other words, $\varphi(P \cap \overline{B}(0, 1)) \subset \mathbb{R}^4 \setminus p^{-1}[B(x, r') \cap P]$.

Define $g : \mathbb{R}^4 \setminus p^{-1}[B(x, r') \cap P] \rightarrow p^{-1}[\partial B(0, 1) \cap P]$ as follows. For $x \in \mathbb{R}^4 \setminus p^{-1}[B(0, 1) \cap P]$, let $g(x)$ be the shortest distance projection of \mathbb{R}^4 onto $p^{-1}[B(0, 1) \cap P]$, and for $y \in [p^{-1}[(B(0, 1) \setminus B(x, r')) \cap P]$, let $g(y)$ be the intersection of $p^{-1}[\partial B(0, 1) \cap P]$ with the ray $[x, y)$ of end point x and passing through y . Then g is continuous.

Notice that $\varphi(P \cap \overline{B}(0, 1)) \subset \mathbb{R}^4 \setminus p^{-1}[B(x, r') \cap P]$, hence $p \circ g \circ \varphi$ is Lipschitz, and sends $P \cap \overline{B}(0, 1)$ continuously to $P \cap \partial B(0, 1)$, with all points of $P \cap \partial B(0, 1)$ fixed. This is impossible. \square

Proof of Proposition 4.8 (3). We know that E_k is the limit of a sequence of deformations $F_l = \varphi_l(P_k)$ of P_k . Then for each l , we have, by Lemma 4.9,

$$(4.11) \quad p_k^i(F_l) \supset p_k^i[\varphi_l(P_k^i \cap \overline{B}(0, 1))] \supset P_k^i \cap \overline{B}(0, 1), i = 1, 2.$$

As a result,

$$(4.12) \quad p_k^i(E_k) \supset P_k^i \cap \overline{B}(0, 1), i = 1, 2.$$

□

We still have to prove (2) and (5) of Proposition 4.8. In fact all we have to do is to prove (5), because it says that E_k is contained in the convex hull C of $P_k \cap \overline{B}(0, 1)$, which gives (2).

Let us verify that $E_k \subset \overline{B}(0, 1)$. First of all we claim that for each $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for all $l > N$,

$$(4.13) \quad F_l \subset B(B(0, 1) \cup P_k, \epsilon),$$

That is to say, F_l is contained in an ϵ -neighborhood of the union of the unit ball and the boundary of U .

In fact, those F_l are uniformly quasiminimal in U , hence are locally uniformly Ahlfors regular in U (c.f. [7] Proposition 4.1). This means that there exists a constant $C > 1$ such that for all l , for all $x \in F_l$ and all $r > 0$ with $B(x, 2r) \subset U$,

$$(4.14) \quad C^{-1}r^2 \leq H^2(F_l \cap B(x, r)) \leq Cr^2.$$

Then if there exists $x \in F_l$ such that $d(x, \overline{B}(0, 1) \cup P_k) > \epsilon$, we should have $B(x, \epsilon) \subset U$, and by the Ahlfors regularity,

$$(4.15) \quad H^2(F_l \cap B(x, \frac{1}{2}\epsilon)) \geq (4C)^{-1}\epsilon^2.$$

Now we deform F_l into $\overline{B}(0, 1)$ by the radial projection π on $\overline{B}(0, 1)$. Here for each l , F_l is a deformation of $P_k \cap \overline{B}(0, 1)$ in U , it means that there exists a compact set $K \subset U$, such that $F_l \setminus K = P_k \cap \overline{B}(0, 1) \setminus K$. The compactness of K gives that $0 < d = d(K, \partial U) = d(K, P_k \setminus B(0, 1))$, and there exists $R > 0$ such that $K \subset B(0, R)$. Set $V = B(0, R) \setminus \overline{B}(P_k \setminus B(0, 1), d)$. Then in V , π is homotopic to the identity map. Now we set $\pi' : U \rightarrow \overline{B}(0, 1) \cap U, \pi'(x) = t(x)x + (1 - t(x))\pi(x)$, where $t(x) = \min\{2d(x, \overline{V})/d, 1\}$. Then π' is a deformation on U , and $\pi'|_V = \pi|_V$, which gives $\pi'(F_l) = \pi(F_l)$. Hence the set $\pi(F_l)$ is a deformation of P_k in U . In other words $\pi(F_l) \in \mathfrak{F}$.

Lemma 4.16. *Let E be rectifiable such that $E \cap B(0, 1 + \epsilon) = \emptyset$. Then*

$$(4.17) \quad H^2(\pi(E)) \leq \frac{1}{(1 + \epsilon)^2} H^2(E).$$

Proof. We are going to prove that π is $\frac{1}{1+\epsilon}$ -Lipschitz on E , which gives (4.17).

Denote by π_ϵ the shortest distance projection on the ball $\overline{B}(0, 1 + \epsilon)$. Then π_ϵ is 1-Lipschitz, and $\pi_\epsilon(\mathbb{R}^4 \setminus B(0, 1 + \epsilon)) \subset \partial B(0, 1 + \epsilon)$. On the other hand, π is $\frac{1}{1+\epsilon}$ -Lipschitz on $\partial B(0, 1 + \epsilon)$. So if $E \subset \mathbb{R}^4 \setminus B(0, 1 + \epsilon)$, then $\pi = \pi \circ \pi_\epsilon$ is $\frac{1}{1+\epsilon}$ -Lipschitz on E . □

Let us return to our sets F_l . We have $B(x, \frac{1}{2}\epsilon) \cap B(0, 1 + \frac{1}{2}\epsilon) = \emptyset$, since $x \notin B(0, 1 + \epsilon)$, and hence

$$\begin{aligned}
(4.18) \quad H^2(\pi(F_l)) &\leq H^2(\pi(F_l \setminus B(x, \frac{1}{2}\epsilon))) + H^2(\pi(F_l \cap B(x, \frac{1}{2}\epsilon))) \\
&\leq H^2(F_l \setminus B(x, \frac{1}{2}\epsilon)) + \frac{1}{(1 + \frac{1}{2}\epsilon)^2} H^2(F_l \cap B(x, \frac{1}{2}\epsilon)) \\
&\leq H^2(F_l) - \frac{4\epsilon + \epsilon^2}{4 + 4\epsilon + \epsilon^2} H^2(F_l \cap B(x, \frac{1}{2}\epsilon)) \\
&\leq H^2(F_l) - \frac{4\epsilon + \epsilon^2}{4 + 4\epsilon + \epsilon^2} \frac{\epsilon^2}{4C} \\
&= H^2(F_l) - C(\epsilon),
\end{aligned}$$

where $C(\epsilon) > 0$ for all $\epsilon > 0$, and $C(\epsilon)$ does not depend on l for l large.

We know that $\{F_l\}$ is a minimizing sequence, therefore for all $\epsilon > 0$, there exists $N > 0$ such that for $l > N$,

$$(4.19) \quad H^2(F_l) \leq \inf_{E \in \mathfrak{F}} H^2(E) + \frac{1}{2}C(\epsilon) < H^2(\pi(F_l)) + C(\epsilon).$$

Hence (4.18) is not true, which means that $F_l \subset B(B(0, 1) \cup P_k, \epsilon) \cap U$, thus we obtain (4.13).

As a result, since E_k is the limit of F_l ,

$$(4.20) \quad E_k \subset \cap_\epsilon B(B(0, 1) \cup P_k, \epsilon) \cap U \subset \overline{B}(0, 1).$$

Lemma 4.21. *Let $C \subset \mathbb{R}^n$ be a closed convex symmetric (with respect to the origin) set with non-empty interior. Then for all $\epsilon > 0$, there exists $\delta > 0$ and a 1-Lipschitz retraction f of \mathbb{R}^n onto C such that f is $1 - \delta$ -Lipschitz on $\mathbb{R}^n \setminus B(C, \epsilon)$.*

Proof. Denote by $\|\cdot\|_C$ the norm on \mathbb{R}^n whose closed unit ball is C . Then there exists $A > 1$ such that

$$(4.22) \quad A^{-1}\|\cdot\|_C \leq \|\cdot\| \leq A\|\cdot\|_C,$$

where $\|\cdot\|$ denotes the Euclidean norm.

For all $a \geq 0$, set $\|\cdot\|_a = \|\cdot\|_C + a\|\cdot\|$ and $C_{a,b}$ the closed ball of radius b under the norm $\|\cdot\|_a$. Then $C_{0,1} = C$. Notice that for all $x \in \mathbb{R}^n \setminus \{0\}$, $\|x\|_a$ is a strictly increasing continuous function of a , $\|\cdot\|_0 = \|\cdot\|_C$, and that $C_{a,b}$ is continuous, decreasing with respect to a and increasing with respect to b , that is,

$$(4.23) \quad C_{a,b} \supset C_{a',b}, \quad C_{a,b} \subset C_{a,b'} \quad \text{for all } a < a', b < b',$$

and

$$(4.24) \quad \bigcap_{a_n \rightarrow a-} C_{a_n,b} = \bigcap_{b_n \rightarrow b+} C_{a,b_n} = C_{a,b}; \quad \bigcup_{a_n \rightarrow a+} C_{a_n,b} = \bigcup_{b_n \rightarrow b-} C_{a,b_n} = C_{a,b}^\circ.$$

Since the norm defined by $C_{a,b}$ contains a part of Euclidean norm, which is uniformly convex, it is easy to verify that

(4.25) for all $a, b > 0$, there exists a constant $M(a, b, A) > 0$, such that
for each $x, y \in \partial C_{a,b}$ with $\alpha_{x,y} < \frac{\pi}{2}$, $B(\frac{x+y}{2}, M(a, b, A)||x-y||^2) \subset C_{a,b}$,

where $\alpha_{x,y} < \pi$ denotes the angle between \vec{Ox} and \vec{Oy} for $x, y \neq 0$, and $B(\frac{x+y}{2}, M(a, b, A)||x - y||^2)$ denotes the euclidean ball centered at $\frac{x+y}{2}$ with radius $M(a, b, A)||x - y||^2$.

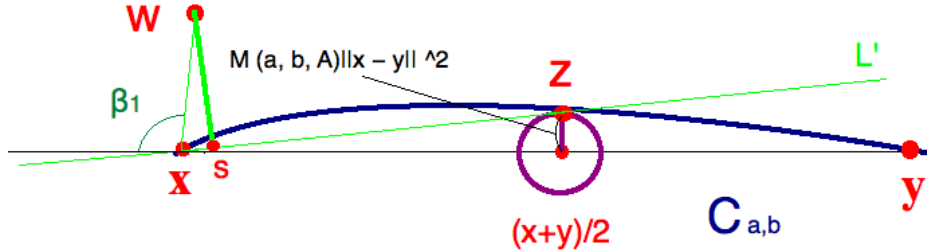
Now for all $\epsilon > 0$, let w, v be two points in $\mathbb{R}^n \setminus B(C_{a,b}, \epsilon)$ such that $\pi_{a,b}(w) = x, \pi_{a,b}(v) = y$, where $\pi_{a,b}$ denotes the shortest distance projection on the convex set $C_{a,b}$. We claim that the angle $\beta_1 \in [0, \frac{\pi}{2}]$ between \vec{xw} and \vec{yx} is smaller than $\arctan \frac{1}{2M(a,b,A)||x-y||}$. (See the picture 4-1 below). In fact, denote by P the plane containing x, y and w , denote by $z \in P$ the point such that $[z, \frac{x+y}{2}] \perp [x, y]$ and $||z - \frac{x+y}{2}|| = M(a, b, A)||x - y||^2$. Then $z \in C_{a,b}$, $[x, z] \in C_{a,b}$, and

$$(4.26) \quad \tan \angle zxy = 2M(a, b, A) \|x - y\|.$$

Then if $\beta_1 > \arctan \frac{1}{2M(a,b,A)||x-y||}$, we have $\angle wxz < \frac{\pi}{2}$. Denote by s the projection of w on L' , the line passing through x and z . Then s is between x and z , or z is between x and s . In both cases $(x, z) \cap (x, s) \neq \emptyset$. Take $x' \in (x, z) \cap (x, s) \subset C_{a,b}$, then $x' \in C_{a,b}$, and $\angle wxz < \angle wx'z$. As a result

$$(4.27) \quad \|w - x\| = \|w - s\| / \sin \angle wxz > \|w - s\| / \sin \angle wx'z = \|w - x'\|,$$

which contradicts the fact that x is the shortest distance projection of w on $C_{a,b}$.



4-1

Similarly we can prove that if β_2 denotes the angle between $\vec{y}\vec{v}$ and $\vec{x}\vec{y}$, then $\beta_2 \leq \arctan \frac{1}{2M(a,b,A)||x-y||}$.

Denote by L the line passing through x and y , and p_L the orthogonal projections on them. Then

$$(4.28) \quad \|w - v\| \geq \|p_L(w) - p_L(v)\| = \|w - x\| \cos \beta_1 + \|x - y\| + \|v - y\| \cos \beta_2.$$

But $w, v \in \mathbb{R}^n \setminus B(C_{a,b}, \epsilon)$, hence we have $\|w - x\| > \epsilon, \|v - y\| > \epsilon$, therefore

$$(4.29) \quad \begin{aligned} \|w - v\| &\geq \|x - y\| + 2\epsilon \cos \arctan \frac{1}{2M(a, b, A)\|x - y\|} \\ &= (1 + \epsilon C(a, b, A))\|x - y\|, \end{aligned}$$

with $C(a, b, A) > 0$.

Notice that (4.29) is true for all pairs of x, y such that $\alpha_{x,y} < \frac{\pi}{2}$. Hence $\pi_{a,b}$ is locally $\frac{1}{1+\epsilon C(a,b,A)}$ -Lipschitz on $\mathbb{R}^n \setminus B(C_{a,b}, \epsilon)$.

Return to the proof of the lemma. Fix an arbitrary $\epsilon > 0$. Then by (4.23) and (4.24), there exists $a, b > 0$ such that

$$(4.30) \quad C \subset C_{a,b} \subset B(C_{a,b}, \frac{\epsilon}{2}) \subset B(C, \epsilon).$$

Now denote by π_C the shortest distance projection on C . Denote by $f = \pi_C \circ \pi_{a,b}$ for a pair of a, b which satisfies (4.30). We want to prove f is the desired map for Lemma 4.21. Since π_C is 1-Lipschitz, for proving the lemma, it is sufficient to prove that $\pi_{a,b}$ is locally $1 - \delta$ -Lipschitz on $\mathbb{R}^n \setminus B(C_{a,b}, \frac{\epsilon}{2})$. Then by (4.29), we take δ such that $1 - \delta = \frac{1}{1 + \frac{1}{2}\epsilon C(a,b,A)}$, and we obtain the conclusion. \square

Corollary 4.31. *Let $E \subset \mathbb{R}^n \setminus B(C, \epsilon)$ be a rectifiable set and f be as in the lemma, then*

$$(4.32) \quad H^d(f(E)) \leq (1 - \delta)^d H^d(E).$$

\square

Now let us return to the proof of Proposition 4.8. Recall that here C is the convex hull of $P_k \cap \overline{B}(0, 1)$.

Proof of (2) and (5) of Proposition 4.8.

We will prove that

$$(4.33) \quad \text{for all } \epsilon > 0, E_k \subset B(C, 2\epsilon).$$

If it is not true, i.e. if $E_k \setminus B(C, 2\epsilon) \neq \emptyset$, then by Ahlfors regularity, $H^2(E_k \setminus B(C, 2\epsilon)) > 0$. We apply Corollary 4.31 to $E_k \setminus B(C, 2\epsilon)$ and the convex set $B(C, \epsilon)$, and obtain that there exists a Lipschitz map f_ϵ of \mathbb{R}^n in $B(C, \epsilon)$, such that

$$(4.34) \quad H^2(f_\epsilon(E_k)) < H^2(E_k),$$

where f_ϵ is as in Lemma 4.21. Then we will get (4.33) if we know that $f_\epsilon(E_k)$ is a deformation of E_k in U .

But we already know that $E_k \subset \overline{B}(0, 1)$. So $W_\epsilon := \{x \in E_k, \pi_\epsilon(x) \neq x\}$ is contained in $\overline{B}(0, 1) \setminus B(C, \epsilon)$, which is compact and far from the boundary of U . Denote by $d(W_\epsilon, \partial U) = d$, and set

$$(4.35) \quad g : \mathbb{R}^n \rightarrow B(C, \epsilon), f(x) = t(x)x + (1 - t(x))f_\epsilon(x), t(x) = \min\{2d(x, W_\epsilon)/d, 1\},$$

then g is a deformation in U , and

$$(4.36) \quad g(E_k) = f_\epsilon(E_k).$$

Therefore, if $E_k \setminus B(C, 2\epsilon) \neq \emptyset$, then g decreases strictly the measure of E_k , which contradicts the fact that E_k is minimal. Hence we have (4.33). But (4.33) is true for all $\epsilon > 0$, hence we have

$$(4.37) \quad E_k \subset C,$$

from which (5) follows. (2) is a direct corollary of (5). This completes the proof of Proposition 4.8. \square

Corollary 4.38. *For each subsequence $\{n_k\}$ such that E_{n_k} converges in $\overline{B}(0, 1)$ for the Hausdorff distance, the limit is $P_0 \cap \overline{B}(0, 1)$.*

Proof. Take such a sequence $\{n_k\}$. Denote by E_∞ the limit of E_{n_k} . We want to apply Theorem 3.1, so we should check its hypotheses for E_∞ .

First, by [4] Thm 4.1, which says that the limit under the Hausdorff distance of a sequence of minimal sets is minimal, we know that E_∞ is also minimal in $B(0, 1)$, because each E_{n_k} is.

Next, (3.2) follows by Proposition 4.8(3) and the fact that E_∞ is the limit of E_{n_k} .

To verify (3.3), if we denote by C_k the convex hull of $P_k \cap \overline{B}(0, 1)$ (and C_0 the convex hull of $P_0 \cap \overline{B}(0, 1)$), then by Proposition 4.8, $E_{n_k} \subset C_{n_k}$. Since E_∞ is the limit of E_{n_k} , for each $m > 0$, there exists $K(m) > 0$ such that for all $k > K(m)$, $E_\infty \subset B(E_{n_k}, \frac{1}{m}) \subset B(C_{n_k}, \frac{1}{m}) \subset \cup_{k=K(m)}^\infty B(C_{n_k}, \frac{1}{m})$. We can also ask that $K(m) > k(l)$ when $m > l$. As a result we have

$$(4.39) \quad E_\infty \subset \cap_{m=1}^\infty \cup_{k=K(m)}^\infty B(C_{n_k}, \frac{1}{m}) = C_0,$$

and therefore

$$(4.40) \quad E_\infty \cap \partial B(0, 1) \subset C_0 \cap \partial B(0, 1) = P_0 \cap \partial B(0, 1).$$

On the other hand, since E_∞ is the limit of the E_{n_k} , it contains $\lim_{k \rightarrow \infty} E_{n_k} \cap \partial B(0, 1) = P_0 \cap \partial B(0, 1)$.

Hence

$$(4.41) \quad E_\infty \cap \partial B(0, 1) = P_0 \cap \partial B(0, 1).$$

which gives (3.3).

For (3.4), Proposition 4.8 (4) gives that $H^2(E_{n_k}) < 2\pi$. But E_{n_k} and E_∞ are minimal sets, by the lower semi continuity of Hausdorff measures for minimal sets ([4] Thm 3.4),

$$(4.42) \quad H^2(E_\infty \cap B(0, 1)) \leq \liminf_k H^2(E_{n_k} \cap B(0, 1)) \leq 2\pi.$$

The equality follows from (3.2), and Lemmas 2.27 and 2.14.

Hence by Theorem 3.1, $E_\infty = P_0 \cap \overline{B}(0, 1)$. □

Notice that $\overline{B}(0, 1)$ is compact, hence there exists a subsequence of $\{E_k\}$ that converges. We still denote this subsequence by $\{E_k\}$ for short. Then by Corollary 4.38, the limit is $P_0 \cap \overline{B}(0, 1)$. So from now on, we will concentrate on this sequence $\{E_k\}_{k>0}$ that converges to $P_0 \cap \overline{B}(0, 1)$.

5 Critical radius

For our sequence $\{E_k\}$, once again, if for each k , E_k is a deformation of P_k , then there should be some sort of pinch at the center, because otherwise, the deformation is injective, and E_k is therefore an essentially disjoint union of images of P_k^1 and P_k^2 . But since P_k^i is minimal, $i = 1, 2$, its image always has a larger measure than P_k^i , therefore E_k is not a better competitor.

Now since we have to pinch, we know that E_k has to get far away from P_k somewhere, and intuitively we have to pay a price for this. In fact, the main issue of the proof is to understand why a pinch costs more than all that it could save for us.

However E_k is not necessarily a deformation of P_k , so what we said just now is just for giving an idea on what we are going to do, that is, to find the place where E_k begins to get away from P_k .

So for ϵ sufficiently small, we want to find a center o , which is not far from the origin, and a scale r , such that E_k is $\epsilon r'$ near some translation of P_k in $B(o, r')$ for all $r' \geq 2r$, but is not ϵr near any translation of P_k in $B(o, r)$. Then this is the place that we are looking for, and r is the critical radius.

As a result, in the small ball $B(o, \frac{1}{4}r)$ where the pinch takes place, probably we cannot see very clearly what happens, so we only control the measure of E_k by an argument of projection, using Corollary 2.45. However, for the part outside the small ball, E_k is near the two planes, and hence is regular, by the regularity property of minimal sets. So we can treat this part more precisely. And finally we will be able to prove that pinching will make us lose more measure on these regular parts than that we can gain in the small ball.

So let's begin by looking for the critical radius in this section, by a stopping time argument.

For each, $k \in \mathbb{N}$ and $i = 1, 2$, set

$$(5.1) \quad C_k^i(x, r) = (p_k^i)^{-1}(B(0, r) \cap P_k^i) + x,$$

where p_k^i is the orthogonal projection on P_k^i , and

$$(5.2) \quad D_k(x, r) = C_k^1(x, r) \cap C_k^2(x, r).$$

So $C_k^i(x, r)$ is a cylinder and $D_k(x, r)$ is the intersection of two cylinders. It is not hard to see that $D_k(x, r) \supset B(x, r)$ and $D_k(0, 1) \cap P_k = B(0, 1) \cap P_k$.

We say that two sets E, F are ϵr near each other in an open set U if

$$(5.3) \quad d_{r,U}(E, F) < \epsilon,$$

where

$$(5.4) \quad d_{r,U}(E, F) = \frac{1}{r} \max\{\sup\{d(y, F) : y \in E \cap U\}, \sup\{d(y, E) : y \in F \cap U\}\}.$$

We set also

$$(5.5) \quad \begin{aligned} d_{x,r}^k(E, F) &= d_{r,D_k(x,r)}(E, F) \\ &= \frac{1}{r} \max\{\sup\{d(y, F) : y \in E \cap D_k(x, r)\}, \sup\{d(y, E) : y \in F \cap D_k(x, r)\}\}. \end{aligned}$$

Remark 5.6. *We should be clear about the fact that*

$$(5.7) \quad d_{r,U}(E, F) \neq \frac{1}{r} d_H(E \cap U, F \cap U).$$

To see this, we can take $U = D_k(x, r)$, and set $E_n = \partial D_k(x, r + \frac{1}{n})$ and $F_n = \partial D_k(x, r - \frac{1}{n})$. Then we have

$$(5.8) \quad d_{x,r}^k(E_n, F_n) \rightarrow 0$$

and

$$(5.9) \quad \frac{1}{r} d_H(E_n \cap D_k(x, r), F_n \cap D_k(x, r)) = \frac{1}{r} d_H(E_n \cap D_k(x, r), \emptyset) = \infty.$$

So $d_{r,U}$ measures rather how the part of one set in the open set U could be approximated by the other set, and vice versa. However we always have

$$(5.10) \quad d_{x,r}^k(E, F) \leq \frac{1}{r} d_H(E \cap D_k(x, r), F \cap D_k(x, r)).$$

Now let us recall that $\{E_k\}$ is a sequence of sets as in Proposition 4.8, with $\theta_k > \frac{\pi}{2} - \frac{1}{k}$, which converges to $P_0 \cap \overline{B}(0, 1)$.

Proposition 5.11. *There exists $\epsilon_0 > 0$, such that if $\epsilon < \epsilon_0$, then for all k large, there exists $r_k \in]0, \frac{1}{2}[$ and $o_k \in B(0, 12\epsilon)$ such that E_k is $2\epsilon r_k$ near $P_k + o_k$ in $D_k(o_k, 2r_k(1 - 12\epsilon))$, but not ϵr_k near $P_k + q$ in $D_k(o_k, r_k)$ for any $q \in \mathbb{R}^4$.*

Remark 5.12. *We will also use the construction for information about intermediate scales in the proof.*

Proof.

We fix ϵ and k , and set $s_i = 2^{-i}$ for $i \geq 0$. Set $D(x, r) = D_k(x, r)$, $d_{x,r} = d_{x,r}^k$ for short.

We proceed in the following way.

Step 1: Denote by $q_0 = q_1 = O$, then in $D(q_0, s_0)$, E_k is ϵs_0 near $P_k + q_1$ if k is large, since $E_k \rightarrow P_0$ and $P_k \rightarrow P_0$ and hence $d_{0,1}(E_k, P_k) \rightarrow 0$.

Step 2: If in $D(q_1, s_1)$ E_k is not ϵs_1 near $P_k + q$ for any q , we stop; if not, there exists a q_2 such that E_k is ϵs_1 near $P_k + q_2$ in $D(q_1, s_1)$. Here we also ask that ϵ be small enough (say, $\epsilon < \frac{1}{100}$) so that $q_2 \in D(q_1, \frac{1}{2}s_1)$, thanks to the conclusion of step 1. Then in $D(q_1, s_1)$, we have simultaneously :

$$(5.13) \quad d_{q_1, s_1}(E_k, P_k + q_1) \leq s_1^{-1} d_{q_0, s_0}(E_k, P_k + q_1) \leq 2\epsilon ; \quad d_{q_1, s_1}(E_k, P_k + q_2) \leq \epsilon.$$

Let us verify that (5.13) implies that $d_{q_1, \frac{1}{2}s_1}(P_k + q_1, P_k + q_2) \leq 12\epsilon$ when ϵ is small, say, $\epsilon < \frac{1}{100}$. In fact, for each $z \in D(q_1, \frac{1}{2}s_1) \cap (P_k + q_1)$, we have $d(z, E_k) \leq d_{q_0, s_0}(E_k, P_k + q_1) \leq \epsilon$, hence there exists $y \in E_k$ such that $d(z, y) \leq \epsilon$. But since $z \in D(q_1, \frac{1}{2}s_1)$, we have $y \in D(q_1, \frac{1}{2}s_1 + \epsilon) \subset D(q_1, s_1)$, and hence $d(y, P_k + q_2) \leq s_1^{-1} d_{q_1, s_1}(E_k, P_k + q_2) \leq 2\epsilon$, therefore $d(z, P_k + q_2) \leq d(z, y) + d(y, P_k + q_2) \leq 3\epsilon$.

On the other hand, suppose $z \in D(q_1, \frac{1}{2}s_1) \cap (P_k + q_2)$, we have $d(z, E_k) \leq s_1^{-1} d_{q_1, s_1}(P_k + q_2, E_k) \leq 2\epsilon$, hence there exists $y \in E_k$ such that $d(z, y) \leq 2\epsilon$. But since $z \in D(q_1, \frac{1}{2}s_1)$, we have $y \in D(q_1, \frac{1}{2}s_1 + 2\epsilon) \subset D(q_0, s_0)$, and hence $d(y, P_k + q_1) \leq d_{q_0, s_0}(E_k, P_k + q_1) \leq \epsilon$, which implies $d(z, P_k + q_1) \leq d(z, y) + d(y, P_k + q_1) \leq 3\epsilon$.

As a result

$$(5.14) \quad d_{q_1, \frac{1}{2}s_1}(P_k + q_1, P_k + q_2) \leq (\frac{1}{2}s_1)^{-1} \times 3\epsilon = 12\epsilon,$$

hence $d_{q_1, \frac{1}{2}s_1}(q_1, q_2) \leq 24\epsilon$, and therefore $d(q_1, q_2) \leq 6\epsilon = 12\epsilon s_1$.

Now we define our iteration process (notice that it depends on ϵ , so we also call it a ϵ -process).

Suppose that all $\{q_i\}_{i \leq n}$ have already been defined, with

$$(5.15) \quad d(q_i, q_{i+1}) \leq 12s_i\epsilon = 12 \times 2^{-i}\epsilon$$

for $0 \leq i \leq n-1$, and hence

$$(5.16) \quad d(q_i, q_j) \leq 24\epsilon s_{\min(i,j)} = 2^{-\min(i,j)} \times 24\epsilon$$

for $0 \leq i, j \leq n$, moreover for all $i \leq n-1$, E_k is ϵs_i near $P_k + q_{i+1}$ in $D(q_i, s_i)$. We say that the process does not stop at step n . In this case

Step n+1 : We look at the situation in $D(q_n, s_n)$.

If E_k is not ϵ near any $P_k + q$ in this ball of radius s_n , we stop, since we have found the $o_k = q_n, r_k = s_n$ as desired. In fact, since $d(q_{n-1}, q_n) \leq 12\epsilon s_{n-1}$, we have $D(q_n, 2s_n(1 - 12\epsilon)) = D(q_n, s_{n-1}(1 -$

$12\epsilon)) \subset D(q_{n-1}, s_{n-1})$, and hence

$$(5.17) \quad \begin{aligned} d_{q_n, 2s_n(1-12\epsilon)}(P_k + q_n, E_k) &\leq (1-12\epsilon)^{-1} d_{q_{n-1}, s_{n-1}}(P_k + q_n, E_k) \\ &\leq \frac{\epsilon}{1-12\epsilon}. \end{aligned}$$

Moreover

$$(5.18) \quad d(o_k, O) = d(q_n, q_1) \leq 2^{-\min(1, n)} \times 24\epsilon = 12\epsilon.$$

Otherwise, we can find a $q_{n+1} \in \mathbb{R}^4$ such that E_k is still ϵs_n near $P_k + q_{n+1}$ in $D(q_n, s_n)$. Then since ϵ is small, $q_{n+1} \in D(q_n, \frac{1}{2}s_n)$. Moreover we have as before $d(q_{n+1}, q_n) \leq 12\epsilon s_n$, and for $i \leq n-1$,

$$(5.19) \quad d(q_i, q_{n+1}) \leq \sum_{j=i}^n d(q_j, q_{j+1}) \leq \sum_{j=i}^n 12 \times 2^{-j} \epsilon \leq 2^{-j} \times 24\epsilon = 2^{-\min(i, n+1)} \times 24\epsilon.$$

Thus we have obtained our q_{n+1} .

Now all we have to do is to prove that for every ϵ small enough, this process has to stop at a finite step. For this purpose, we are going to estimate the measure of E_k . So we need the lemma below.

Lemma 5.20. *There exists $\epsilon_0 > 0$, such that for all $\epsilon < \epsilon_0$, k large enough, and for every n such that the ϵ -process does not stop before n (which means in particular that there exists $q_n \in B(q_{n-1}, \frac{1}{2}s_{n-1})$ such that E_k is ϵs_{n-1} near $P_k + q_n$ in $D(q_{n-1}, s_{n-1})$),*

$$(5.21) \quad E_k \cap (D(0, 1) \setminus D(q_n, s_n)) = F_n^1 \cup F_n^2$$

where F_n^1, F_n^2 do not meet each other. Moreover

$$(5.22) \quad P_k^i \cap (D(0, 1) \setminus D(q_n, s_n)) \subset p_k^i(F_n^i)$$

where p_k^i is the orthogonal projection on $P_k^i, i = 1, 2$.

We will prove a more general version of this lemma in the next section (Proposition 6.1 (2)). So let us admit it for the moment.

Since $H^2(E_k) < 2\pi$, there exists $n_k > 0$ such that

$$(5.23) \quad \inf_{q \in \mathbb{R}^4} H^2(P_k \setminus D(q, s_{n_k})) > H^2(E_k).$$

Then our process has to stop before step n_k , because otherwise by the above lemma, we have the disjoint decomposition

$$(5.24) \quad E_k = [E_k \cap D(q_{n_k}, s_{n_k})] \uplus F_{n_k}^1 \uplus F_{n_k}^2,$$

and hence

$$(5.25) \quad \begin{aligned} H^2(E_k) &\geq H^2(F_{n_k}^1) + H^2(F_{n_k}^2) \geq H^2[p_k^1(F_{n_k}^1)] + H^2[p_k^2(F_{n_k}^2)] \\ &\geq H^2(P_k \setminus D(q_{n_k}, s_{n_k})) > H^2(E_k). \end{aligned}$$

This is impossible. □

6 Projection properties and regularity of E_k

As we have said in the previous section, the next step is to give some useful properties of E_k , including the regularity for the flat part of E_k out of the small critical ball, and the surjectivity of the projections of E_k inside the ball. This is the main aim of this section. It gives also the proof of Lemma 5.20.

We are sorry that the proof for (3) and (4) of Proposition 6.1 is surprisingly painful. We have to derive the property from the geometric construction of the proof for Theorem 4.1, which is already complicated. But we did not find any easier proof.

So let us first state the main proposition that we will prove in this section.

Proposition 6.1. *There exists $\epsilon_0 > 0$, such that for all $\epsilon < \epsilon_0$ and k large, if the ϵ -process does not stop before the step n , then*

(1) $E_k \cap (D_k(0, \frac{39}{40}) \setminus D_k(q_n, \frac{1}{10}s_n))$ is composed of two disjoint pieces $G^i, i = 1, 2$, such that:

$$(6.2) \quad G^i \text{ is the graph of a } C^1 \text{ map } g^i : D_k(0, \frac{39}{40}) \setminus D_k(q_n, \frac{1}{10}s_n) \cap P_k^i \rightarrow P_k^{i\perp}$$

with

$$(6.3) \quad \|\nabla g^i\|_\infty < 1;$$

(2) (A more general version of Lemma 5.20) for every $\frac{1}{10}s_n \leq t \leq s_n$,

$$(6.4) \quad E_k \cap (D_k(0, 1) \setminus D_k(q_n, t)) = G_t^1 \cup G_t^2$$

where G_t^1, G_t^2 do not meet each other. Moreover

$$(6.5) \quad P_k^i \cap (D_k(0, 1) \setminus C_k^i(q_n, t)) \subset p_k^i(G_t^i)$$

where p_k^i is the orthogonal projection on $P_k^i, i = 1, 2$;

(3) for each $\frac{1}{10}s_n < t < s_n$, there exists a sequence $\{F_l^n(t) = f_l^n(t)((P_k + q_n) \cap \overline{D}_k(q_n, t + \frac{1}{l}))\}_{l \geq 1}$ of deformations of $(P_k + q_n) \cap \overline{D}_k(q_n, t + \frac{1}{l})$ in $\overline{D}_k(q_n, t + \frac{1}{l})$, with

$$(6.6) \quad f_l^n(t)((P_k + q_n) \cap \partial C_k^i(q_n, t + \frac{1}{l})) \subset \partial C_k^i(q_n, t + \frac{1}{l}),$$

that converge to $E_k \cap D_k(q_n, t)$ in $D_k(0, 1)$;

(4) the orthogonal projections $p_k^i : E_k \cap D_k(q_n, t) \rightarrow P_k^i \cap C_k^i(q_n, t), i = 1, 2$ are surjective, for all $\frac{1}{10}s_n \leq t \leq s_n$.

In order to prove (1) we will apply a regularity theorem on varifolds. First we give some useful notations below.

$G(n, d)$ is the Grassmann manifold $G(\mathbb{R}^n, d)$;

for every $T \in G(n, d)$, we denote by π_T the orthogonal projection on the d -plane represented by T ;

for every measure ν on \mathbb{R}^n , $\theta^d(\nu, x) = \lim_{r \rightarrow 0} \frac{\nu B(a, r)}{\alpha(d)r^d}$ (if the limit exists) is the density of ν on x , where $\alpha(d)$ denote the volume of the d -dimensional unit ball;

$\mathbb{V}_d(\mathbb{R}^n)$ denote the set of all d -varifold in \mathbb{R}^n , i.e. all Radon measures on $G_d(\mathbb{R}^n) = \mathbb{R}^n \times G(n, d)$;

for each $V \in \mathbb{V}_d(\mathbb{R}^n)$, $\|V\|$ is the Radon measure on \mathbb{R}^n such that for each $A \subset \mathbb{R}^n$, $\|V\|(A) = V(G_d(\mathbb{R}^n) \cap \{(x, S) : x \in A\})$;

$\delta(V)$ denotes the first variation of V , that is, the linear map from $\mathfrak{X}(\mathbb{R}^n)$ to \mathbb{R} , defined by

$$(6.7) \quad \delta V(g) = \int Dg(x) \cdot \pi_S dV(x, S)$$

for $g \in \mathfrak{X}(\mathbb{R}^n)$. Here $\mathfrak{X}(\mathbb{R}^n)$ is the vector space of all C^∞ maps from \mathbb{R}^n to \mathbb{R}^n with compact support.

In our case, we are only interested in rectifiable varifolds. In fact, with each d -rectifiable set E we associate a d -varifold, denoted by V_E , in the following sense : for each $B \subset \mathbb{R}^n \times G(n, d)$, we have

$$(6.8) \quad V_E(B) = H^d\{x : (x, T_x E) \in B\}.$$

Recall that $T_x E$ is the d -dimensional tangent plane of E at x , it exists for almost all $x \in E$, because E is d -rectifiable. Then $\|V_E\| = H^d|_E$. Moreover, the density $\theta^d(\|V_E\|, x)$ exists for almost all $x \in E$.

Theorem 6.9 (c.f.[1] Regularity theorem at the beginning of section 8). *Suppose $2 \leq d < p < \infty$, $q = \frac{p}{p-1}$. Corresponding to every ϵ with $0 < \epsilon < 1$ there is $\eta > 0$ with the following property:*

Suppose $0 < R < \infty$, $0 < \mu < \infty$, $V \in \mathbb{V}_d(\mathbb{R}^n)$, $a \in \text{spt}\|V\|$ and

1) $\theta^d(\|V\|, x) \geq \mu$ for $\|V\|$ almost all $x \in B(a, R)$;

2) $\|V\|B(a, R) \leq (1 + \eta)\mu\alpha(d)R^d$;

3) $\delta V(g) \leq \eta\mu^{\frac{1}{p}}R^{\frac{d}{p-1}}(\int |g|^q \mu \|V\|)^{\frac{1}{q}}$ whenever $g \in \mathfrak{X}(\mathbb{R}^n)$ and $\text{spt } g \subset B(a, R)$.

Then there are $T \in G(n, d)$ and a continuously differentiable function $F : T \rightarrow \mathbb{R}^n$, such that $\pi_T \circ F = 1_T$,

$$(6.10) \quad \|DF(y) - DF(z)\| \leq \epsilon(|y - z|/R)^{1 - \frac{d}{p}} \text{ whenever } y, z \in T,$$

and

$$(6.11) \quad B(a, (1 - \epsilon)R) \cap \text{spt}\|V\| = B(a, (1 - \epsilon)R) \cap \text{image } F.$$

Remark 6.12. *1) In the theorem, since $\pi_T \circ F = 1_T$, we can see that F is in fact the graph of a C^1 function f , defined by $f(t) = \pi_{T^\perp} F(t)$, with $t \in T$, π_{T^\perp} the orthogonal projection on the orthogonal space T^\perp of T . Moreover $\|Df(t)\| \leq \|DF(t)\|$ for all $t \in T$.*

2) If E is a locally minimal set, then V_E is stationary, i.e. $\delta V_E = 0$. Hence the condition 3) is automatically true. In fact if we set $g_t(x) = (1 - t)x + tg(x)$, then

$$(6.13) \quad \delta V_E(g) = \frac{d}{dt} H^d(g_t(E \cap \text{spt} g)),$$

which can be deduced from the area formula. Thus if E is minimal, $\delta V_E = 0$.

Proposition 6.14. *For all $n > d > 0$, there exists $\epsilon_1 = \epsilon_1(n, d) > 0$ such that the following is true. Let E be a locally minimal set of dimension d in an open set $U \subset \mathbb{R}^n$, with $U \supset B(0, 2)$ and $0 \in E$. Then if E is ϵ_1 near a d -plane P in $B(0, 1)$, then E coincides with the graph of a C^1 map $f : P \rightarrow P^\perp$ in $B(0, \frac{3}{4})$. Moreover $\|\nabla f\|_\infty < 1$.*

Proof. We will prove it only for $d = 2$. Proofs for other dimensions are similar.

First let us verify the conditions in Theorem 6.9, with $a = 0, R = 1, \mu = 1, \eta < \frac{1}{10}$, and ϵ small, to choose later.

1) Since E is minimal, for each $x \in E$, the density of E at x is at least 1, hence 1) is true.

2) We know that E is ϵ_1 near a 2-dimensional affine plane P in $B(0, 1)$, and $H^2(P \cap B(0, 1)) \leq \alpha(2) = \pi$. Then by Lemma 16.43 in [5], we can choose ϵ_1 (which depends on ϵ , since η depends on ϵ) such that 2) is true. In particular

$$(6.15) \quad H^2(B(0, 1) \cap E) \leq \frac{11}{10}\pi.$$

3) comes from the minimality of E , by Remark 6.12 2), with any $p > 2$.

Then when p is sufficiently large, by Theorem 6.9, there exists a plane T and a C^1 map F from T to \mathbb{R}^4 such that in $B(0, 1 - \epsilon)$, E coincides with the image of F , and

$$(6.16) \quad \|DF(y) - DF(z)\| \leq \epsilon(|y - z|/R)^{1-\frac{2}{p}} \leq \left(\frac{7}{4}\right)^{1-\frac{2}{p}} \epsilon \leq 2\epsilon$$

for all $y, z \in F^{-1}(E \cap B(0, 1 - \epsilon))$.

Notice that for each $y \in F^{-1}(E \cap B(0, 1 - \epsilon)) \subset T$, $DF(y)(T)$ is the tangent plane of E at $F(y)$. Thus (6.16) means that the tangent planes of E do not vary a lot in $B(0, 1 - \epsilon)$.

Now denote by Q the plane parallel to P (the plane in the statement of Proposition 6.14) and passing by the origin, π the projection on Q , and π' the projection on Q^\perp . We claim that for each $y \in T$ such that $F(y) \in B(0, 1 - \epsilon)$ and each $u \in T_y E = DF(y)(T)$,

$$(6.17) \quad \|\pi(u)\| \geq \frac{3}{4}\|u\|.$$

In fact, if (6.17) is not true, then there exists $y \in F^{-1}(E \cap B(0, 1 - \epsilon))$ and $u \in T_y E$ such that $\|\pi(u)\| < \frac{3}{4}\|u\|$. Denote by $t = DF(y)^{-1}(u) \in T$, then

$$(6.18) \quad \|\pi(DF(y)(t))\| \leq \frac{3}{4}\|DF(y)(t)\|.$$

By (6.16), for all $z \in F^{-1}(E \cap B(0, 1 - \epsilon))$,

$$\begin{aligned}
(6.19) \quad & \|\pi(DF(z)(t))\| \leq \|\pi \circ (DF(z) - DF(y))(t)\| + \|\pi \circ DF(y)(t)\| \\
& \leq 2\epsilon\|t\| + \frac{3}{4}\|DF(y)(t)\| \\
& \leq 2\epsilon\|t\| + \frac{3}{4}\|DF(y)(t) - DF(z)(t)\| + \frac{3}{4}\|DF(z)(t)\| \\
& \leq 2\epsilon\|t\| + \frac{3}{2}\epsilon\|t\| + \frac{3}{4}\|DF(z)(t)\| \\
& = \frac{7}{2}\epsilon\|t\| + \frac{3}{4}\|DF(z)(t)\|.
\end{aligned}$$

But $\pi_T \circ F = 1_T$ implies that for all $t \in T$,

$$(6.20) \quad \|DF(z)(t)\| \geq \|\pi_T \circ DF(z)(t)\| = \|t\|,$$

therefore

$$(6.21) \quad \|\pi(DF(z)(t))\| \leq \left(\frac{7}{2}\epsilon + \frac{3}{4}\right)\|DF(z)(t)\|.$$

So when ϵ is sufficiently small, we have

$$(6.22) \quad \|\pi' \circ DF(z)(t)\| \geq \frac{1}{2}\|DF(z)(t)\|$$

for all $z \in F^{-1}(E \cap B(0, 1 - \epsilon))$.

Set $e_1 = \pi' \circ DF(0)(t) / \|\pi' \circ DF(0)(t)\|$ a unit vector in Q . Then we have

$$(6.23) \quad \langle e_1, DF(0)(t) \rangle = \frac{1}{2}\|DF(0)(t)\|.$$

Then for all $z \in F^{-1}(E \cap B(0, 1 - \epsilon))$, still by (6.16) and (6.20),

$$\begin{aligned}
(6.24) \quad & \langle e_1, DF(z)(t) \rangle = \langle e_1, (DF(z) - DF(0))(t) \rangle + \langle e_1, DF(0)(t) \rangle \\
& \geq \langle e_1, DF(0)(t) \rangle - \|\langle e_1, (DF(z) - DF(0))(t) \rangle\| \\
& \geq \frac{1}{2}\|DF(0)(t)\| - 2\epsilon\|t\| \\
& \geq \frac{1}{2}[\|DF(z)(t)\| - \|(DF(0) - DF(z))(t)\|] - 2\epsilon\|t\| \\
& \geq \frac{1}{2}[\|DF(z)(t)\| - 2\epsilon\|t\|] - 2\epsilon\|t\| \geq \frac{1}{2}\|DF(z)(t)\| - 3\epsilon\|DF(z)(t)\| \\
& \geq \frac{1}{3}\|DF(z)(t)\|.
\end{aligned}$$

As a result, if we take $z \in F^{-1}(E \cap \partial B(0, 1 - \epsilon))$, such that $\vec{z} = \lambda t$ with $\lambda > 0$, we have

$$\begin{aligned}
(6.25) \quad & \langle e_1, F(z) - F(0) \rangle = \langle e_1, \int_0^1 DF(sz)(t)ds \rangle = \int_0^1 \langle e_1, DF(sz)(t) \rangle ds \\
& \geq \int_0^1 \frac{1}{3}\|DF(sz)(t)\|ds = \frac{1}{3} \int_0^1 \|DF(sz)(t)\|ds \\
& \geq \frac{1}{3}\|F(z) - F(0)\| = \frac{1}{3}\|F(z)\| = \frac{1}{3}(1 - \epsilon),
\end{aligned}$$

which implies that when ϵ_1, ϵ are sufficiently small, there exists no translation $Q + x$ of Q (including P) such that $B_{0,1}(Q + x, E) < \epsilon_1$. Contradiction.

So we have (6.17). In other words, $D\pi$ is always injective in $E \cap B(0, 1 - \epsilon)$. Then by the implicit function theorem, for all $x \in E \cap B(0, 1 - \epsilon)$, there exists $r_x > 0$ and $g_x : Q \rightarrow Q^\perp$ such that in $\pi^{-1}[B(\pi(x), r_x) \cap Q] \cap B(x, 2r_x)$, E coincides with the graph of g_x on $B(\pi(x), r_x)$. Moreover by (6.17)

$$(6.26) \quad \|\nabla g_x(x)\| \leq 1.$$

Next let us verify that

$$(6.27) \quad \pi(E \cap B(0, 1 - \epsilon)) \supset Q \cap B(0, \frac{3}{4}).$$

Recall that E is ϵ_1 near a plane P parallel to Q in $B(0, 1)$, hence $E \cap B(0, 1) \subset B(P, \epsilon_1)$ and $d(0, P) \leq \epsilon_1$, therefore $d(Q, P) \leq \epsilon_1$. Hence $E \cap B(0, 1 - \epsilon) \subset B(Q, 2\epsilon_1)$, such that $E \cap \partial B(0, 1 - \epsilon) \subset B(Q, 2\epsilon)$, and therefore

$$(6.28) \quad \text{for every } x \in E \cap \partial B(0, 1 - \epsilon), \|\pi(x)\| \geq \sqrt{(1 - \epsilon)^2 - (2\epsilon)^2} \geq \frac{3}{4}.$$

But by Theorem 6.9, $E \cap B(0, 1 - \epsilon)$ is a topological disc, hence by a topological argument similar to that of Lemma 4.9, (6.28) gives $\pi(E \cap B(0, 1 - \epsilon)) \supset B(0, \frac{3}{4}) \cap Q$. Thus we have (6.27).

Now let Γ be a connected component of $F := E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$. Then it is both open and closed in F . But we know that $D\pi(x)$ is injective for all $x \in F$, so π is an open map, such that $\pi(\Gamma)$ is open in $B(0, \frac{3}{4}) \cap Q$. On the other hand, we claim that $\pi(\Gamma)$ is also closed in $B(0, \frac{3}{4}) \cap Q$. In fact, suppose that $\{x_n\} \subset \pi(\Gamma)$ is a sequence of points that converge to a point $x_\infty \in B(0, \frac{3}{4}) \cap Q$. For each n , take $y_n \in \Gamma$ such that $\pi(y_n) = x_n$. Then $\{y_n\} \subset \bar{\Gamma}$, which is compact, hence the sequence $\{y_n\}$ admits a limit point $y_\infty \in \bar{\Gamma}$. Therefore we have $\pi(y_\infty) = x_\infty$. We want to prove that $y_\infty \in \Gamma$, hence we look at $\bar{\Gamma} \setminus \Gamma$. Since Γ is closed in $F = E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$,

$$(6.29) \quad \begin{aligned} \bar{\Gamma} \setminus \Gamma &\subset E \cap \partial[B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)] \\ &= E \cap \{[\partial B(0, 1 - \epsilon) \cap \pi^{-1}(\bar{B}(0, \frac{3}{4}) \cap Q)] \cup [\partial(\pi^{-1}(B(0, \frac{3}{4}) \cap Q)) \cap \bar{B}(0, 1 - \epsilon)]\}. \end{aligned}$$

We know that the distance $d(\partial B(0, 1 - \epsilon) \cap \pi^{-1}(\bar{B}(0, \frac{3}{4}) \cap Q), Q) > \sqrt{(1 - \epsilon)^2 - (\frac{3}{4})^2} > \sqrt{\frac{7}{16}} - 2\epsilon$, and $d(P, Q) \leq d(0, P) < \epsilon_1$, since $0 \in Q$ and $0 \in E$. As a result,

$$(6.30) \quad d(\partial B(0, 1 - \epsilon) \cap \pi^{-1}(\bar{B}(0, \frac{3}{4}) \cap Q), P) > \sqrt{\frac{7}{16}} - 2\epsilon - \epsilon_1 > \epsilon_1$$

when ϵ and ϵ_1 are both small. Then the hypothesis says that for each $y \in E \cap B(0, 1)$, $d(y, P) < \epsilon_1$, hence $[E \cap B(0, 1)] \cap \partial B(0, 1 - \epsilon) \cap \pi^{-1}(\bar{B}(0, \frac{3}{4}) \cap Q) = \emptyset$. As a result, $\bar{\Gamma} \subset E \cap B(0, 1)$ does not meet $\partial B(0, 1 - \epsilon) \cap \pi^{-1}(\bar{B}(0, \frac{3}{4}) \cap Q)$. On combining with (6.29),

$$(6.31) \quad \bar{\Gamma} \setminus \Gamma \subset \partial(\pi^{-1}(B(0, \frac{3}{4}) \cap Q)) \cap \bar{B}(0, 1 - \epsilon).$$

But after the hypothesis, $\pi(y_\infty) = x_\infty \in B(0, \frac{3}{4}) \cap Q$, hence $y_\infty \notin \partial(\pi^{-1}(B(0, \frac{3}{4}) \cap Q))$. Therefore, $y_\infty \notin \bar{\Gamma} \setminus \Gamma$. Hence $y_\infty \in \Gamma$, and thus $x_\infty \in \pi(\Gamma)$.

So $\pi(\Gamma)$ is also closed in $B(0, \frac{3}{4}) \cap Q$. As a result, $\pi(\Gamma) = B(0, \frac{3}{4}) \cap Q$.

Now we claim that

$$(6.32) \quad \pi : \Gamma \rightarrow B(0, \frac{3}{4}) \cap Q \text{ is a covering space of } Q \cap B(0, \frac{3}{4}).$$

In fact, since E is compact, the continuous map $\pi : E \rightarrow Q$ is proper, such that for each $x \in Q \cap B(0, \frac{3}{4})$, $\pi^{-1}(x) \cap \Gamma$ is a finite set. Denote this set $\{y_1, \dots, y_N\}$. Then by the conclusion before (6.26), for each $1 \leq j \leq N$, there exists $r_j > 0$ such that in $\pi^{-1}[B(x, r_j) \cap Q] \cap B(y_j, 2r_j)$, Γ coincides with the graph of a map $g_j : Q \rightarrow Q^\perp$ on $B(x, r_j) \cap Q$. Set $r = \min_j r_j$, then $\pi^{-1}[B(x, r) \cap Q]$ contains the finite disjoint union of these $g_j(B(x, r) \cap Q)$, $1 \leq j \leq N$. On the other hand, for each $y \in \pi^{-1}(B(x, r) \cap Q)$, take a connected component γ of $\pi^{-1}(B(x, r) \cap Q)$ such that $y \in \gamma$, then by an argument similar to the one for Γ on $B(0, \frac{3}{4}) \cap Q$ above, $\pi(\gamma) \supset B(x, r) \cap Q$, in particular, there exists $1 \leq j \leq N$ such that $y_j \in \gamma$. But in this case we have $\gamma = g_j(B(x, r) \cap Q)$, hence $y \in g_j(B(x, r) \cap Q)$. As a result, $\pi^{-1}[B(x, r) \cap Q]$ is just a finite disjoint union of $g_j(B(x, r) \cap Q)$, $1 \leq j \leq N$, and on each of these pieces, π is an homeomorphism of $B(x, r)$, where (6.32) follows.

But $Q \cap B(0, \frac{3}{4})$ is simply connected, Γ is its connected covering space by π , hence π has to be a homeomorphism. Then by the conclusion around (6.26), Γ is the graph of a C^1 map from Q to Q^\perp whose gradient is of L^∞ norm less than 1. In particular, the measure of Γ is larger than the measure of $B(0, \frac{3}{4}) \cap Q$, which is $\frac{9}{16}\pi$.

Now denote by $\Gamma_1, \dots, \Gamma_n, \dots$ all the connected components of $F = E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$, the measure of each Γ_i is larger than $\frac{9}{16}\pi$. Then if there exists more than one Γ_i , we have

$$(6.33) \quad H^2(E \cap B(0, 1)) > H^2(E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)) \geq 2 \times \frac{9}{16}\pi = \frac{9}{8}\pi,$$

which contradicts (6.15).

Thus $E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$ is a trivial covering space of $Q \cap B(0, \frac{3}{4})$. Combine with (6.27), $\pi^{-1} : B(0, \frac{3}{4}) \cap Q \rightarrow E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$ is a C^1 map, which coincides with g_x around $\pi(x)$ for each $x \in E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$, and hence $\|\nabla g\|_\infty < 1$. Now set $f = g \circ \pi : P \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q) \rightarrow E \cap B(0, 1 - \epsilon) \cap \pi^{-1}(B(0, \frac{3}{4}) \cap Q)$. Thus since P is parallel to Q , we get the desired conclusion. \square

Remark 6.34. For $d = 2, n = 4$, we can also get the same result by Theorem 1.15 of [6], without all those complicated concepts such as varifolds, etc.

Corollary 6.35. There exists $\epsilon_1 > 0$ such that if k is large enough, E is a locally minimal sets in a domain $U \subset \mathbb{R}^4$, $D_k(0, 1) \subset U$, and E is ϵ_1 near a plane P in $D_k(0, 1)$, then in $D_k(0, \frac{1}{2})$, E coincides with the graph of a C^1 map $f : P \rightarrow P^\perp$. Moreover $\|\nabla f\|_\infty < 1$.

Proof. When k is large enough, we have $D_k(0, \frac{1}{2}) \subset B(0, \frac{3}{4}) \subset B(0, 1) \subset D_k(0, 1)$. Then if E is ϵ_1 near a plane P in $D_k(0, 1)$, this implies that E is ϵ_1 near P in $B(0, 1)$. Thus by Proposition 6.14, in $B(0, \frac{3}{4})$, E is the graph of a C^1 map $C^1 f : P \rightarrow P^\perp$, with $\|f\|_\infty < 1$. Therefore in $D_k(0, \frac{1}{2})$, too. \square

Now fix a large enough k , and denote by $D(x, r) = D_k(x, r)$, $C^i(x, r) = C_k^i(x, r)$ for $i = 1, 2$, and $d_{x,r} = d_{x,r}^k$.

Proof of (1) of Proposition 6.1.

Since k is large, P_k is very near P_0 , there exists $0 < \epsilon_3 < \frac{1}{100}$ (which does not depend on k for k large), such that for all $\epsilon < \epsilon_3$, if $x \in \mathbb{R}^4$ and E is any set such that $d_{x,r}(E, P_k + q) < \epsilon$, with $q \in \mathbb{R}^4$ and $d(x, q) < 20\epsilon r$, then in $D(x, r) \setminus D(q, \frac{1}{100}r)$, E is the disjoint union of two pieces E^1, E^2 , such that in $D(x, r)$ minus a small hole, E^1 is near $P_k^1 + q$, but far from $P_k^2 + q$, and vice-versa for E^2 . More precisely,

$$(6.36) \quad E^i \subset B((P_k^i + q) \cap D(x, r) \setminus D(q, \frac{1}{100}r), \epsilon r)$$

and

$$(6.37) \quad \begin{aligned} d(B(P_k^1 \cap D(x, r) \setminus D(q, \frac{1}{100}r), \epsilon r), B(P_k^2 \cap D(x, r), \epsilon r)) &> \frac{1}{80}r; \\ d(B(P_k^2 \cap D(x, r) \setminus D(q, \frac{1}{100}r), \epsilon r), B(P_k^1 \cap D(x, r), \epsilon r)) &> \frac{1}{80}r. \end{aligned}$$

In particular,

$$(6.38) \quad d(E^1, E^2) \geq \frac{1}{80}r.$$

Take $\epsilon_0 = \min\{\frac{1}{2}\epsilon_3, \frac{1}{40}\epsilon_1, 10^{-5}\}$. Then for every $\epsilon < \epsilon_0$, if the ϵ -process does not stop before the step n , E_k is ϵs_{n-1} near $P_k + q_n$ in $D(q_{n-1}, s_{n-1})$. Fix this n , and denote by $q = q_n, x = q_{n-1}, r = s_{n-1}$ for short. Then since $\epsilon < \epsilon_3$,

$$(6.39) \quad E_k \cap D(x, r) \setminus D(q, \frac{1}{100}r) \text{ is the disjoint union of } E^1, E^2 \text{ such that (6.36)-(6.38) hold.}$$

$$\text{For each } y \in E^1 \cap D(x, r - \frac{1}{40}r) \setminus D(q, \frac{1}{20}r) = E^1 \cap D(q_{n-1}, \frac{39}{40}s_{n-1}) \setminus D(q_n, \frac{1}{10}s_n),$$

$$(6.40) \quad d_{y, \frac{1}{40}r}(E_k, P_k + q) < 40\epsilon < \epsilon_1,$$

where by definition,

$$(6.41) \quad \begin{aligned} d_{y, \frac{1}{40}r}(E_k, P_k + q) &= \frac{1}{40r} \max\{\sup\{d(z, P_k + q) : z \in E_k \cap D(y, \frac{1}{40}r)\}, \\ &\quad \sup\{d(z, E_k) : z \in P_k + q \cap D(y, \frac{1}{40}r)\}\}. \end{aligned}$$

For the second term,

$$(6.42) \quad \sup\{d(z, E_k) : z \in (P_k + q) \cap D(y, \frac{1}{40}r)\} \geq \sup\{d(z, E_k) : z \in (P_k^1 + q) \cap D(y, \frac{1}{40}r)\}$$

since $P_k^1 \subset P_k$; for the first term, notice first that

$$(6.43) \quad \begin{aligned} & \sup\{d(z, P_k + q) : z \in E_k \cap D(y, \frac{1}{40}r)\} \\ &= \sup\{d(z, (P_k + q) \cap D(y, \frac{1}{40}r + \epsilon r)) : z \in E_k \cap D(y, \frac{1}{40}r)\} \end{aligned}$$

since we already know that $d_{y, \frac{1}{40}r}(E_k, P_k + q) < 40\epsilon$, which implies that for each $z \in E_k \cap D(y, \frac{1}{40}r)$, there exists $w \in P_k + q$ such that $d(z, w) < \epsilon r$. Set $W = \{w \in P_k + q, d(z, w) < \epsilon r\}$. Then

$$(6.44) \quad d(z, P_k + q) = d(z, W).$$

For all $w \in W$,

$$(6.45) \quad \begin{aligned} w &\in (P_k + q) \cap D(y, \frac{1}{40}r + \epsilon r) \subset (P_k + q) \cap D(y, \frac{1}{40}r + \frac{1}{100}r) \\ &\subset (P_k + q) \cap D(x, r) \setminus D(q, \frac{1}{100}r) \\ &= [(P_k^1 + q) \cap D(x, r) \setminus D(q, \frac{1}{100}r)] \cup (P_k^2 + q) \cap D(x, r) \setminus D(q, \frac{1}{100}r). \end{aligned}$$

Then w has to belong to $(P_k^1 + q) \cap D(x, r) \setminus D(q, \frac{1}{100}r)$, because otherwise

$$(6.46) \quad z \in B(w, \epsilon r) \cap E_k \subset B((P_k^2 + q) \cap D(x, r) \setminus D(q, \frac{1}{100}r), \epsilon r) \cap E_k = E^2$$

by (6.36) and (6.37), which contradicts the fact that $z \in E^1$. Hence

$$(6.47) \quad d(z, P_k + q) = d(z, W) \geq d(z, P_k^1 + q) \text{ for } z \in E_k \cap D(y, \frac{1}{40}r),$$

therefore

$$(6.48) \quad \sup\{d(z, P_k + q) : z \in E_k \cap D(y, \frac{1}{40}r)\} \geq \sup\{d(z, P_k^1 + q) : z \in E_k \cap D(y, \frac{1}{40}r)\}.$$

Add (6.42) and (6.48) together we obtain

$$(6.49) \quad d_{y, \frac{1}{40}r}(E^1, P_k^1 + q) \leq d_{y, \frac{1}{40}r}(E^1, P_k + q) < 40\epsilon < \epsilon_1.$$

Now $P_k^1 + q$ is a plane, hence we can use Corollary 6.35, which gives

$$(6.50) \quad \begin{aligned} & \text{for each } y \in E^1 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r), \text{ in } D(y, \frac{1}{80}r), \\ & E_k \text{ is the graph of a } C^1 \text{ map } f_y : P_k^1 \rightarrow P_k^{1\perp} \text{ with } \|\nabla f_y\| < 1. \end{aligned}$$

But $E_k \cap D(y, \frac{1}{80}r) = E^1 \cap D(y, \frac{1}{80}r)$, which implies that around every point $y \in E^1 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$, E^1 is locally a C^1 graph on P_k^1 .

Let us verify that in $D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$, E^1 coincides with the graph of a C^1 map on the whole P_k^1 , whose gradient is of norm L^∞ less than 1. However we have already our small local graph near every point, with small gradient, so we only have to show that the projection $p_k^1 : E^1 \rightarrow P_k^1 \cap C^1(x, \frac{39}{40}r) \setminus C^1(q, \frac{1}{20}r)$ is bijective on $E^1 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$.

Surjectivity: Set $A = p_k^1(E^1) \cap C^1(x, \frac{39}{40}r) \setminus C^1(q, \frac{1}{20}r)$. Then A is non empty. We are going to show that $A = P_k^1 \cap C^1(x, \frac{39}{40}r) \setminus C^1(q, \frac{1}{20}r)$.

First A is closed in $P_k^1 \cap C^1(x, \frac{39}{40}r) \setminus C^1(q, \frac{1}{20}r)$, since E^1 is compact in $D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$.

But A is also open, because if $z \in A$, then there exists $y \in E^1 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$ such that $p_k^1(y) = z$. Thus by (6.50), we know that $B(z, \frac{1}{80}r) \cap P_k^1 \subset A$. Hence A is open.

Notice that $P_k^1 \cap C^1(x, \frac{39}{40}r) \setminus C^1(q, \frac{1}{20}r)$ is connected, hence $A = P_k^1 \cap C^1(x, \frac{39}{40}r) \setminus C^1(q, \frac{1}{20}r)$, which gives the surjectivity.

Injectivity: Suppose p_k^1 is not injective. Then there exists $y_1, y_2 \in E^1 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$ such that $p_k^1(y_1) = p_k^1(y_2)$. In other words

$$(6.51) \quad y_1 - y_2 \in P_k^{1\perp}.$$

We know that in $D(y_1, \frac{1}{80}r)$, E^1 is a graph, hence $y_2 \notin D(y_1, \frac{1}{80}r)$. In other words, $|y_1 - y_2| > \frac{1}{80}r$. Hence there exists at least one point between y_1, y_2 whose distance to $P_k^1 + q$ is larger than $\frac{1}{160}r > \epsilon r$. This gives a contradiction with (6.48) and the fact that $d(z, P_k + q) \leq rd_{x,r}(E_k, P_k + q) < \epsilon r$.

Therefore $p_k^i(q)$ is injective. Denote by f^1 the map defined on $P_k^1 \cap C^1(x, \frac{39}{40}r + \frac{r}{80}) \setminus C^1(q, \frac{r}{20} - \frac{r}{80})$ and which coincides with f_y on every $B(p_k^1(y), \frac{1}{80}r) \cap P_k^1$; then in $E^1 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$, E^1 is the graph of f^1 with $\|\nabla f^1\|_\infty < 1$.

By a similar argument we obtain also that $E^2 \cap D(x, \frac{39}{40}r) \setminus D(q, \frac{1}{20}r)$ is the graph of a C^1 map f^2 , which sends $P_k^2 \cap C^2(x, \frac{39}{40}r) \setminus C^2(q, \frac{1}{20}r)$ in $P_k^{2\perp}$. Recall that $D(x, r) = D(q_{n-1}, s_{n-1})$, and by replacing E^i by $E^i(n)$, f^i by $f^i(n)$, we can obtain our graph $E^i(n) = f^i(n)(P_k^i \cap C^i(q_{n-1}, \frac{39}{40}s_{n-1}) \setminus C^i(q_n, \frac{1}{10}s_n))$, on condition that the ϵ process does not stop at the step n . But of course if it does not stop at step n , it does not stop at any step before n neither. Therefore for all $j \leq n$, we have decompositions of $E_k \cap D(q_{j-1}, \frac{39}{40}s_{j-1}) \setminus D(q_j, \frac{1}{10}s_j)$ as disjoint unions

$$(6.52) \quad E_k \cap D(q_{j-1}, \frac{39}{40}s_{j-1}) \setminus D(q_j, \frac{1}{10}s_j) = E^1(j) \cup E^2(j),$$

and

$$(6.53) \quad E^i(j) \text{ is the graph of } g^i(j) \text{ on } P_k^i \cap C^i(q_{j-1}, \frac{39}{40}s_{j-1}) \setminus C^i(q_j, \frac{1}{10}s_j).$$

We can easily verify that if j, l are such that $x \in P_k^i \cap [C^i(q_{j-1}, \frac{39}{40}s_{j-1}) \setminus C^i(q_j, \frac{1}{10}s_j)] \cap [C^i(q_{l-1}, \frac{39}{40}s_{l-1}) \setminus C^i(q_l, \frac{1}{10}s_l)]$, then $g^i(j)(x) = g^i(l)(x) \in E^i(j) \cap E^i(l)$. Hence set

$$(6.54) \quad \begin{aligned} g^i &: P_k^i \cap D(0, \frac{39}{40}) \setminus D(p_k^i(q_n), \frac{1}{10}s_n) \rightarrow P_k^{i\perp}; \\ g^i(x) &= g^i(j)(x) \text{ on } P_k^i \cap D(p_k^i(q_{j-1}), \frac{39}{40}s_{j-1}) \setminus D(p_k^i(q_j), \frac{1}{10}s_j), 1 \leq j \leq n; \end{aligned}$$

then $\|\nabla g^i\|_\infty < 1$, and its graph is $G^i = [\cup_{j=0}^n E^i(j)] \cap D(0, \frac{39}{40}) \setminus D(q_n, \frac{1}{10}s_n)$.

Thus all we have to do is to show that G^1, G^2 are disjoint. This is equivalent to saying that for $0 \leq j, l \leq n$, $E^1(j) \cap E^2(l) = \emptyset$. This is true for $j = l$, so suppose that $j < l$. Then for

all point $x \in E^1(j)$ there are two cases: either $x \in D(q_{l-1}, \frac{39}{40}s_{l-1}) \setminus D(q_l, \frac{1}{10}s_l)$, either not. In the second case, $x \notin E^2(l)$ automatically because $E^2(l) \subset D(q_{l-1}, \frac{39}{40}s_{l-1}) \setminus D(q_l, \frac{1}{10}s_l)$; in the first case, $x \in E_k \cap D(q_{l-1}, \frac{39}{40}s_{l-1}) \setminus D(q_l, \frac{1}{10}s_l) = E^1(l) \cup E^2(l)$. Then $D(q_{l-1}, s_{l-1}) \setminus D(q_{j-1}, \frac{1}{10}s_{j-1}) \neq \emptyset$ implies that $l - j \leq 4$.

But $x \in E^1(j)$ implies that $d(x, P_k^2 + q_j) > \frac{1}{80}s_{j-1}$ because of (6.37). Hence

$$(6.55) \quad d(x, P_k^2 + q_l) > \frac{1}{80}s_{j-1} - d(q_j, q_l).$$

While by (5.16) we have $d(q_j, q_l) \leq 24\epsilon \times 2^{-\min(l,j)} = 24\epsilon_0 \times 2^{-j} \leq \frac{48}{10^5}s_{j-1}$, hence

$$(6.56) \quad d(x, P_k^2 + q_l) > \frac{3}{400}s_{j-1} \geq 2^4 \frac{3}{400}s_{l-1} > \epsilon_{s_{l-1}},$$

therefore $x \notin E^2(l)$, because of (6.36).

Hence for all $0 \leq j, l \leq n$, $E^1(j) \cap E^2(l) = \emptyset$, therefore $G^1 \cap G^2 = \emptyset$. Thus we complete the proof of (1). \square

Proof of (2) of Proposition 6.1. Since E_k is ϵ near P_k in $D(0, 1)$, we can see that $E_k \cap D(0, 1) \setminus D(0, \frac{1}{100})$ is the disjoint union of two pieces E^1, E^2 which satisfy (6.36)-(6.38). By (1), we know that in $D(0, \frac{39}{40}) \setminus D(q_n, \frac{1}{10}s_n)$ E_k is composed of two disjoint graphs G^1, G^2 on $P_k^1 \cap C^1(0, \frac{39}{40}) \setminus C^1(q_n, \frac{1}{10}s_n)$ and $P_k^2 \cap C^2(0, \frac{39}{40}) \setminus C^2(q_n, \frac{1}{10}s_n)$ respectively. Then for $\frac{1}{10}s_n < t < s_n$, $G_t^i = E^i \cup G^i \setminus D(q_n, t)$, hence (6.4) is true. Moreover, since $G^i, i = 1, 2$ are graphs, we have

$$(6.57) \quad p_k^i(G^i \setminus D(q_n, t)) \supset P_k^i \cap D(0, \frac{39}{40}) \setminus C_k^i(q_n, t),$$

therefore (6.5) is also true if we replace $D(0, 1)$ by $D(0, \frac{39}{40})$, because $G^i \setminus D(q_n, t) \subset G_t^i$.

So we just have to prove

$$(6.58) \quad P_k^i \cap D(0, 1) \setminus D(0, \frac{39}{40}) \subset p_k^i(G_t^i)$$

We prove it for $i = 1$ for example. We know that E_k is ϵ near P_k in $D(0, 1)$. Hence by (6.36),

$$(6.59) \quad E^2 \subset B(P_k^2 \cap D(0, 1) \setminus D(0, \frac{1}{100}), \epsilon) \subset B(P_k^2, \frac{1}{100})$$

However when k is large, we have $p_k^1(P_k^2 \cap D(0, 1)) \subset P_k^1 \cap D(0, \frac{1}{5})$, and therefore

$$(6.60) \quad p_k^1(E^2) \subset P_k^1 \cap D(0, \frac{1}{4}).$$

On the other hand, we know that

$$(6.61) \quad p_k^1(E_k \cap D(0, \frac{1}{100})) \subset P_k^1 \cap D(0, \frac{1}{100})$$

hence we have

$$(6.62) \quad p_k^1(E_k \setminus E^1) = p_k^1[E^2 \cup (E_k \cap D(0, \frac{1}{100}))] \subset D(0, \frac{1}{4}) \cap P_k^1.$$

By Proposition 4.8, $p_k^1(E_k) \supset P_k^1 \cap D(0, 1)$, so we get that

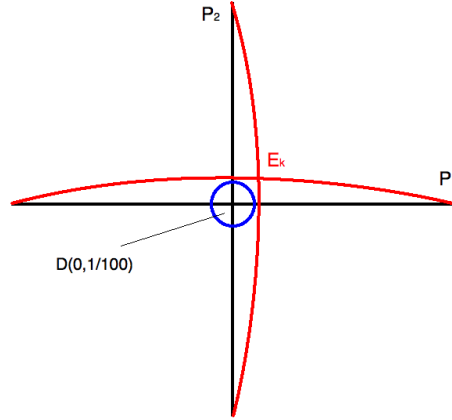
$$(6.63) \quad p_k^1(E^1) \supset P_k^1 \cap D(0, 1) \setminus D(0, \frac{1}{4}).$$

Therefore

$$(6.64) \quad \begin{aligned} p_k^1(G_t^1) &= p_k^1(G^1 \setminus D(q_n, t)) \cup p_k^1(E^1) \\ &\supset [P_k^1 \cap D(0, \frac{39}{40}) \setminus D(q_n, t)] \cup [P_k^1 \cap D(0, 1) \setminus D(0, \frac{1}{4})] = p_k^1 \cap D(0, 1) \setminus D(q_n, t), \end{aligned}$$

where (6.5) follows for $i = 1$. The proof for $i = 2$ is similar. \square

Now let us deal with (3) and (4). First of all we give a small remark. In fact all we want is 4), i.e., the surjectivity of the projections, to estimate the measure of E_k for the part where we do not know much about its structure. Notice that by the proof of Theorem 4.1, we know that E_k is the limit of a sequence $\{H_l\}$ of deformations of P_k in $U = \mathbb{R}^4 \setminus [P_k \setminus B(0, 1)]$. Hence in E the projection p_k^i of E_k is automatically surjective on $D(0, 1) \cap P_k^i$. Next we look into $D(0, \frac{1}{2})$, since E_k is near P_k , the part of $E_k \setminus D(0, \frac{1}{2})$ which is near P_k^1 has no projection in $D(0, \frac{1}{2}) \cap P_k^1$, and the part of $E_k \setminus D(0, \frac{1}{2})$ which is near P_k^2 has a very small projection on P_k^1 , and this projection is very near the origin, hence we can say that outside a small ball, say $D(0, \frac{1}{100})$, the projection p_k^1 from $E_k \cap D(0, \frac{1}{2})$ to P_k^1 is surjective on $P_k^1 \cap D(0, \frac{1}{2}) \setminus D(0, \frac{1}{100})$. However in $D(0, \frac{1}{100})$, we can not say directly that the projection of E_k comes from $E_k \cap D(0, \frac{1}{2})$, because part of it may come from the part near P_k^2 . (Picture 6-1 below may give an idea).



6-1

Therefore the idea to prove 4) is still to prove that $E_k \cap \overline{D}(0, \frac{1}{2})$ is the limit of a sequence of deformations of $P_k \cap \overline{D}(0, \frac{1}{2})$. In other words, we can contract the part outside $\overline{D}(0, \frac{1}{2})$ into $\overline{D}(0, \frac{1}{2})$. This part is very flat and regular, hence intuitively such a contraction will not change essentially the structure of E_k in $D(0, \frac{1}{2})$.

Here we still have to say, if E_k is itself a deformation of P_k and is very near P_k , then we can easily contract the part $E_k \setminus D(0, \frac{1}{2})$ to $E_k \cap \partial D(0, \frac{1}{2})$, just like we contract an annulus to the interior circle,

because $E_k \setminus D(0, \frac{1}{2})$ is roughly composed of two pieces of C^1 graph on $P_k^i \setminus D(0, \frac{1}{2})$, thanks to (6.2). Then next we can carry on the same operation in $D(0, \frac{1}{2})$, if E_k is still near some translation of P_k in $D(0, \frac{1}{2})$. At last, we will arrive at the scale where the ϵ -process stops. Hence we can say that $E_k \cap D(q_n, t)$ is a deformation of P_k , too, because it is a deformation of E_k .

However E_k may not be a deformation of P_k . Hence we will use the fact that E_k is the limit of a sequence of deformations $\{H_l\}$, and we want to apply the argument above to prove that $H_l \cap \overline{D}(0, \frac{1}{2})$ is a deformation of $P_k \cap \overline{D}(0, \frac{1}{2})$. But this time, H_l is not minimal, hence we cannot use (6.2) to say that $H_l \cap \partial D(0, \frac{1}{2})$ is a very regular curve, and therefore it is not that easy to manage to contract H_l directly on $H_l \cap \partial D(0, \frac{1}{2})$.

Then what we are going to do is, first use the shortest distance projection π to project H_l on $\overline{D}(0, \frac{1}{2})$. Then the points of $H_l \setminus D(0, \frac{1}{2})$ are sent by π to $\partial D(0, \frac{1}{2})$ (but the image is not $H_l \cap \partial D(0, \frac{1}{2})$ anymore). To continue to project $\pi(H_l) \setminus D(q_2, \frac{1}{4})$ in $\overline{D}(q_2, \frac{1}{4})$, we use the fact that $\pi(H_l) \setminus D(q_2, \frac{1}{4})$ is composed of two disjoint pieces near $P_k^1 + q_2$ and $P_k^2 + q_2$ respectively. To guarantee this, by the argument before (6.36), $\pi(H_l)$ should be $\epsilon_3 s_2 = \frac{1}{2} \epsilon_3$ near $P_k + q_2$. For the part inside $D(0, \frac{1}{2})$, there is no problem, because the ϵ -process does not stop here. However for the part on the boundary $\partial D(0, \frac{1}{2})$, things are complicated because this part contains the projection of the $H_l \setminus D(0, \frac{1}{2})$, which is $2\epsilon_3 s_2$ near $p_k + q_2$, not $\epsilon_3 s_2$ near it. But luckily we can manage first to do a contraction in $\partial D(0, \frac{1}{2})$, to contract everything near $P_k + q_2$, without moving points inside $D(0, \frac{1}{2})$. Thus we make things more complicated, but at last we can still carry on with the proof.

After all these, we can prove the surjectivity of projections of $E_k \cap D(q_n, t)$, since it is the limit of a sequence of deformations.

Now we begin to concretize the above idea.

Proof of (3) and (4) of Proposition 6.1. Fix a $\frac{1}{10} s_n \leq t \leq s_n$.

We know that for all $j \leq n$, in every $D(q_{j-1}, s_{j-1})$, E_k is ϵs_{j-1} near $P_k + q_j$, because the ϵ -process does not stop on step j . Then set E_k is the limit of H_l , hence we can suppose that l is large, so that in every $D(q_{j-1}, s_{j-1})$, H_l is $2\epsilon s_{j-1}$ near $P_k + q_j$, for all $j \leq n$. By definition of ϵ_0 , we have $2\epsilon < 2\epsilon_0 < \epsilon_3$, hence (6.36) and (6.37) give that in $D(q_{j-1}, s_{j-1}) \setminus D(q_j, \frac{1}{100} s_{j-1})$, H_l is the union of two disjoint pieces H_l^1, H_l^2 with

$$(6.65) \quad H_l^i \subset B(P_k^i + q_j \cap D(q_{j-1}, s_{j-1}) \setminus D(q_j, \frac{1}{100} s_{j-1}), 2\epsilon r).$$

We will construct our deformation F_l^n by recurrence on $j < n$.

For $j = 1$, define $\pi_1 : H_l \rightarrow D(q_1, s_1)$ the shortest distance projection from \mathbb{R}^4 to $D(q_1, s_1)$. Notice that although $H_l \cap D(q_1, s_1)$ is $2\epsilon s_1$ near $P_k^1 + q_2$ in $D(q_1, s_1)$, $\pi_1(H_l \setminus D(q_1, s_1))$ is not necessarily $2\epsilon s_1$ near $P_k^1 + q_2$ in $D(q_1, s_1)$. Therefore we will modify it a little, to be able to continue decompose it into two disjoint pieces that verify some conditions similar to (6.36)-(6.38).

By (6.36), in $D(0,1) \setminus D(q_1, s_1)$, H_l is the union of two disjoint pieces H_l^1, H_l^2 , where H_l^i is very near P_k^i , so we have

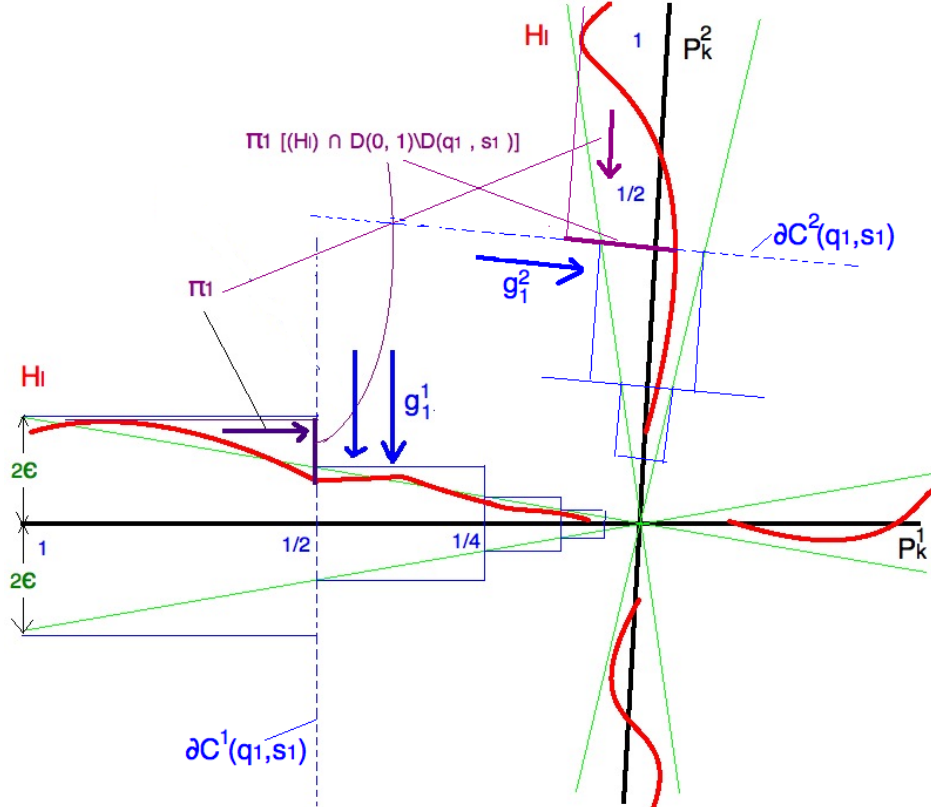
$$(6.66) \quad \pi_1(H_l^i) \cap D(0,1) \setminus D(q_1, s_1) \subset \partial C^i(q_1, s_1) \cap B(P_k^i, 2\epsilon).$$

As a result, the image by π_1 of each H_l^i outside $D(q_1, s_1)$ is contained in the cylinder $C^i(q_1, s_1)$, and also contained in a very small neighborhood of the plane P_k^i . I.e., it is contained in a 3-dimensional thin band around $P_k^i \cap \partial C^i(q_1, s_1)$. In particular, it is far from the boundary $\partial C^j(q_1, s_1)$ for $i \neq j$. Therefore we can define g_1 on $\pi_1(H_l)$ by

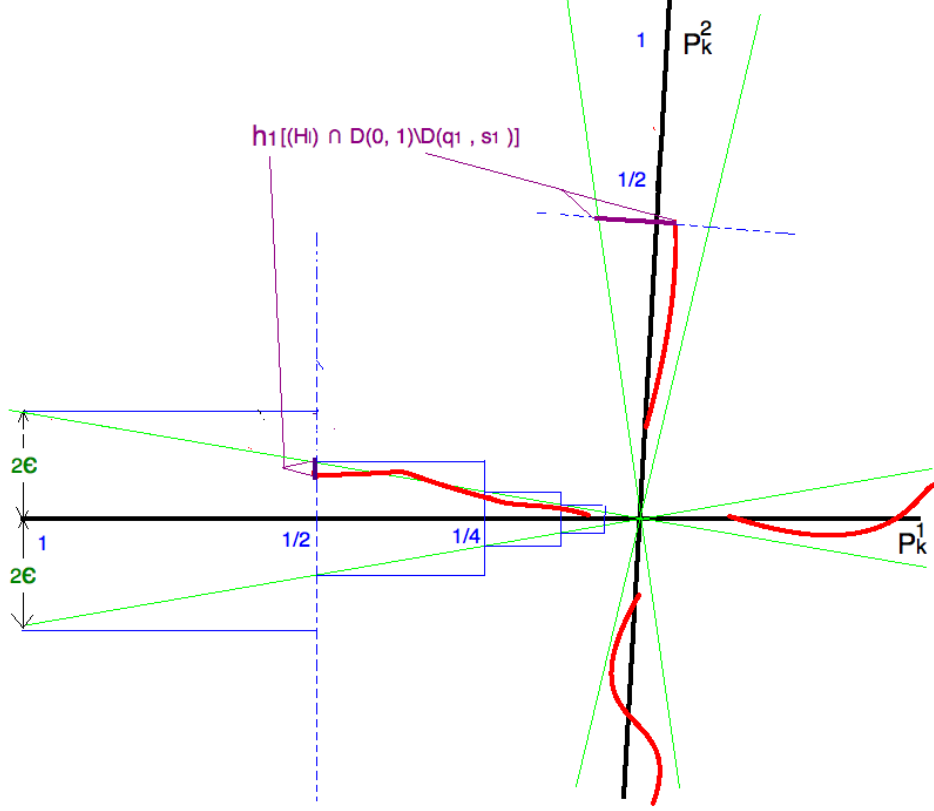
$$(6.67) \quad g_1(x) = \begin{cases} x & ; x \in D(q_1, s_1) ; \\ g_1^i(x) & ; x \in \partial C^i(q_1, s_1). \end{cases}$$

where g_1^i is the orthogonal projection on $B(P_k^i + q_2, 2\epsilon s_1)$.

It is clear that for all points $x \in H_l \cap D(q_1, s_1) \subset \pi_1(H_l)$, neither π_1 nor g_1^i can move it, since in $D(q_1, s_1)$ H_l is $2\epsilon s_1$ near $P_k^1 + q_2$. Hence the action of g_1^i is just to press all points on the boundary $\partial C_k^i(q_1, s_1)$ into a $2s_1\epsilon$ neighborhood of P_k^i , without leaving the boundary. Then g_1 is 2-Lipschitz (locally 1-Lipschitz). Set $h_1 = g_1 \circ \pi_1$, then h_1 is 2-Lipschitz, and moreover $h_1(H_l) \subset \overline{D}(q_1, s_1)$ is $2\epsilon s_1$ near P_k . See Picture 6-2 and 6-3 below. 6-2 is the set, H_l , and in 6-3 we give the image of H_l after h_1 .



6-2



6-3

Now we have defined h_1 in $D(0, 1) = D(q_0, s_0)$, which deforms H_l into $D(q_1, s_1)$ and keeps the image $2\epsilon s_1$ near $P_k + q_2$. But h_1 is 2-Lipschitz and is defined on a compact set H_l , hence we can extend it on the whole \mathbb{R}^4 . Still call the obtained function h_1 . Then it is still 2-Lipschitz.

Now suppose that for j , we have constructed a 2-Lipschitz deformation h_j which deform H_l into $D(q_j, s_j)$ and whose image is $2\epsilon s_j$ near $P_k + q_{j+1}$ in $D(q_j, s_j)$. If $j < n - 1$, then we can define π_{j+1} the shortest distance projection to $D(q_{j+1}, s_{j+1})$, and then similarly we project all point on the boundary of $C^i(q_{j+1}, s_{j+1})$ into the $2\epsilon s_{j+1}$ neighborhood of $P_k^i + q_{j+2}$. Denote this projection by g_{j+1} , and set $h_{j+1} = g_{j+1} \circ \pi_{j+1} \circ h_j$. Then h_{j+1} is a 2-Lipschitz deformation that maps H_l in $D(q_{j+1}, s_{j+1})$ and its image is ϵs_{j+1} near $P_k + q_{j+2}$ in $D(q_{j+1}, s_{j+1})$. Thus we obtain our h_{j+1} .

Therefore by recursion, we can define the h_j until h_{n-1} .

Now for each l , we will define a deformation $h_n(l)$, which will deform H_l into $D(q_n, t + \frac{1}{l})$. Denote by $p_n(l)$ the shortest distance projection on $D(q_n, t + \frac{1}{l})$, and for $x \in \overline{C^i}(q_n, t + \frac{1}{l}) \setminus C^i(q_n, t)$, set $g_n^i(l)(x) = (id, g^i) \circ p_k^i \circ p_n(l)$, where g^i is as in (6.2). Then we define the deformation $h_n(l)$ of

$P_k \cap D(0, 1)$ as follows. For $x \in P_k \cap D(0, 1)$:

$$(6.68) \quad h_n(l)(x) = \begin{cases} h_{n-1}(x), & h_{n-1}(x) \in D(q_n, t) ; \\ sg_n^i(l) \circ h_{n-1}(x) + (1-s)h_{n-1}(x), & h_{n-1}(x) \in \partial C^i(q_n, t + \frac{1}{l}s). \end{cases}$$

Then we can see that $h_n(l)(P_k) \cap \overline{D}(q_n, t) = H_l \cap \overline{D}(q_n, t)$, and $h_n(l)(P_k) \cap \partial D(q_n, t + \frac{1}{l}) = E_k \cap \partial D(q_n, t + \frac{1}{l})$. Between $\overline{D}(q_n, t)$ and $\partial D(q_n, t + \frac{1}{l})$ the image of $h_n(l)$ is the image of a homotopy between H_l and E_k . Then since H_l converges to E_k , $F_l^n(t) = h_n(l)(P_k)$ converges to $E_k \cap D(q_n, t)$ when l goes to infinity.

Denote by $a_l(t)$ the affine deformation which sends $P_k \cap B(0, 1)$ to $(P_k + q_n) \cap D(q_n, t + \frac{1}{l})$, and set

$$(6.69) \quad f_l^n(t) = h_n(l) \circ a_l(t).$$

Then $F_l^n(t) = f_l^n(t)((P_k + q_n) \cap \overline{D}(q_n, t + \frac{1}{l}))$ verifies all conditions in (3).

The condition (4) is an immediate corollary of (3). In fact we know that $F_l^n(t)$ is a deformation of $(P_k + q_n) \cap D(q_n, t + \frac{1}{l})$ which sends $\partial C^i(q_n, t)$ in $\partial C^i(q_n, t)$, hence

$$(6.70) \quad p_k^i(F_l^n(t)) \supset P_k^i \cap C_k^i(q_n, t + \frac{1}{l}).$$

Therefore since $E_k \cap D(q_n, t)$ is the limit of $F_l^n(t)$, each projection p_k^i is surjective from $E_k \cap D(q_n, t)$ to $P_k^i \cap C_k^i(q_n, t)$. \square

7 Argument of harmonic extension

In this section we will give some fundamental estimates on the measure of the graph of a C^1 function on an almost concentric annulus. One can easily skip this section and admit these estimates, and continue the proof of Theorem 1.2 from the beginning of the next section.

Proposition 7.1. *Suppose $0 < r_0 < \frac{1}{2}$ and $u_0 \in C^1(\partial B(0, r_0) \cap \mathbb{R}^2, \mathbb{R})$. Denote by $m(u_0) = \frac{1}{2\pi r_0} \int_{\partial B(0, r_0)} u_0$ its average.*

Then for all $u \in C^1((\overline{B(0, 1)} \setminus B(0, r_0)) \cap \mathbb{R}^2, \mathbb{R})$ that satisfies

$$(7.2) \quad u|_{\partial B(0, r_0)} = u_0$$

we have

$$(7.3) \quad \int_{B(0, 1) \setminus B(0, r_0)} |\nabla u|^2 \geq \frac{1}{4} r_0^{-1} \int_{\partial B(0, r_0)} |u_0 - m(u_0)|^2.$$

Proof.

Let u be a C^1 function as in the statement of Proposition 7.1. We define $\tilde{u} \in C(\overline{B(0, \frac{1}{r_0})} \setminus B(0, r_0)) \cap \mathbb{R}^2, \mathbb{R})$, which is also C^1 except for on $\partial B(0, 1)$, by

$$(7.4) \quad \tilde{u}(x) = \begin{cases} u(x), & x \in \overline{B(0, 1)} \setminus B(0, r_0); \\ u(\frac{x}{|x|^2}), & x \in \overline{B(0, \frac{1}{r_0})} \setminus B(0, 1). \end{cases}$$

Then we have

$$(7.5) \quad \int_{\overline{B(0, \frac{1}{r_0})} \setminus B(0, r_0)} |\nabla \tilde{u}|^2 = 2 \int_{\overline{B(0, 1)} \setminus B(0, r_0)} |\nabla u|^2,$$

because $t \mapsto \frac{x}{|x|^2}$ is a conformal map, which does not change Dirichlet's energy.

Now since \tilde{u} satisfies the boundary condition

$$(7.6) \quad \tilde{u}(x)|_{\partial B(0, r_0)} = u_0(x), \quad \tilde{u}(x)|_{\partial B(0, \frac{1}{r_0})} = u_0(\frac{x}{|x|^2}),$$

its Dirichlet's energy $\int_{\overline{B(0, \frac{1}{r_0})} \setminus B(0, r_0)} |\nabla \tilde{u}|^2$ is no less than that of the harmonic function v satisfying the same boundary condition. So we are going to calculate $\int_{\overline{B(0, \frac{1}{r_0})} \setminus B(0, r_0)} |\nabla v|^2$.

We write

$$(7.7) \quad u(r_0, \theta) = u_0(\theta) = m(u_0) + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta),$$

and set

$$(7.8) \quad \begin{aligned} v : \overline{B(0, \frac{1}{r_0})} \setminus B(0, r_0) &\rightarrow \mathbb{R}, \\ v(r, \theta) &= m(u_0) + \sum_{n=1}^{\infty} a_n(r^n + r^{-n}) \cos n\theta + \sum_{n=1}^{\infty} b_n(r^n + r^{-n}) \sin n\theta. \end{aligned}$$

Then v is harmonic.

The boundary condition

$$(7.9) \quad v(x)|_{\partial B(0, r_0)} = u_0(x), \quad v(x)|_{\partial B(0, \frac{1}{r_0})} = u_0(\frac{x}{|x|^2})$$

gives that

$$(7.10) \quad a_n(r_0^n + r_0^{-n}) = A_n, \quad b_n(r_0^n + r_0^{-n}) = B_n.$$

For estimate ∇v , write

$$(7.11) \quad \begin{aligned} v(r, \theta) &= m(u_0) + \sum_1^{\infty} a_n(r^n + r^{-n}) \cos n\theta + \sum_1^{\infty} b_n(r^n + r^{-n}) \sin n\theta \\ &= m(u_0) + \sum_1^{\infty} \frac{r^n + r^{-n}}{r_0^n + r_0^{-n}} (A_n \cos n\theta + B_n \sin n\theta) \\ &=: m(u_0) + \sum_1^{\infty} v_n(r, \theta). \end{aligned}$$

It is not hard to verify that we can differentiate v term by term. Hence we have

$$(7.12) \quad \frac{\partial v}{\partial \theta} = \sum_{n=1}^{\infty} n(r^n + r^{-n})(-a_n \sin n\theta + b_n \cos n\theta)$$

and

$$(7.13) \quad \frac{\partial v}{\partial r} = \sum_{n=1}^{\infty} n(r^{n-1} - r^{-n-1})(a_n \cos n\theta + b_n \sin n\theta).$$

Therefore

$$(7.14) \quad \begin{aligned} |\nabla v|^2 &= \left| \frac{\partial v}{\partial r} \right|^2 + \left| \frac{1}{r} \frac{\partial v}{\partial \theta} \right|^2 \\ &= \sum_n n^2 (r^{2n-2} + r^{-2n-2})(a_n^2 + b_n^2) - \frac{2}{r^2} \sum_n n^2 (a_n^2 \cos 2n\theta - b_n^2 \cos 2n\theta + 2a_n b_n \sin 2n\theta) \\ &\quad + 2 \sum_{n < m} nm (r^{n-1} r^{m-1} + r^{-n-1} r^{-m-1}) \\ &\quad \{a_n a_m \cos(n-m)\theta + a_n b_m \sin(m-n)\theta + a_m b_n \sin(n-m)\theta - b_n b_m \cos(n-m)\theta\} \\ &\quad - 2 \sum_{n < m} nm (r^{n-1} r^{-m-1} + r^{-n-1} r^{m-1}) \\ &\quad \{a_n a_m \cos(n+m)\theta - b_n b_m \cos(n+m)\theta + a_n b_m \sin(n+m)\theta + b_n a_m \sin(n+m)\theta\}. \end{aligned}$$

Note that for $n \neq m$ and $m, n \geq 1$,

$$(7.15) \quad \begin{aligned} \int_0^{2\pi} \cos(n-m)\theta d\theta &= \int_0^{2\pi} \sin(n-m)\theta d\theta = \int_0^{2\pi} \cos(n+m)\theta d\theta \\ &= \int_0^{2\pi} \sin(n+m)\theta d\theta = \int_0^{2\pi} \cos 2n\theta d\theta = \int_0^{2\pi} \sin 2n\theta d\theta = 0, \end{aligned}$$

therefore

$$(7.16) \quad \int_0^{2\pi} |\nabla v|^2 d\theta = 2\pi \sum_n n^2 (r^{2n-2} + r^{-2n-2})(a_n^2 + b_n^2),$$

hence

$$(7.17) \quad \begin{aligned} \int_{B(0, \frac{1}{r_0}) \setminus B(0, r_0)} |\nabla v|^2 &= \int_{r_0}^{\frac{1}{r_0}} r dr \int_0^{2\pi} |\nabla v|^2 d\theta \\ &= \int_{r_0}^{\frac{1}{r_0}} r dr \cdot 2\pi \sum_n n^2 (r^{2n-2} + r^{-2n-2})(a_n^2 + b_n^2) \\ &= 2\pi \sum_n n(a_n^2 + b_n^2)(r_0^{-n} - r_0^n)(r_0^n + r_0^{-n}). \end{aligned}$$

But for $r_0 < \frac{1}{2}$ and $n \geq 1$, we have

$$(7.18) \quad r_0^{-n} - r_0^n \geq \frac{1}{2}(r_0^n + r_0^{-n})$$

and hence

$$\begin{aligned}
(7.19) \quad \int_{B(0, \frac{1}{r_0}) \setminus B(0, r_0)} |\nabla v|^2 &\geq 2\pi \sum_n n(a_n^2 + b_n^2) \frac{1}{2}(r_0^n + r_0^{-n})^2 \\
&= \pi \sum_n n(A_n^2 + B_n^2) \geq \pi \sum_n (A_n^2 + B_n^2).
\end{aligned}$$

But

$$(7.20) \quad \pi \sum_{n=1}^{\infty} (A_n^2 + B_n^2) = \frac{1}{2} \int_0^{2\pi} |u_0(\theta) - m(u_0)|^2 d\theta = \frac{1}{2} r_0^{-1} \int_{\partial B(0, r_0)} |u_0(s) - m(u_0)|^2 ds.$$

Therefore

$$(7.21) \quad \int_{B(0, \frac{1}{r_0}) \setminus B(0, r_0)} |\nabla v|^2 \geq \frac{1}{2} r_0^{-1} \int_{\partial B(0, r_0)} |u_0(s) - m(u_0)|^2 ds.$$

Now return to the function u . We have

$$\begin{aligned}
(7.22) \quad \int_{B(0, 1) \setminus B(0, r_0)} |\nabla u|^2 &= \frac{1}{2} \int_{B(0, \frac{1}{r_0}) \setminus B(0, r_0)} |\nabla \tilde{u}|^2 \\
&\geq \frac{1}{2} \int_{B(0, \frac{1}{r_0}) \setminus B(0, r_0)} |\nabla v|^2 \geq \frac{1}{4} r_0^{-1} \int_{\partial B(0, r_0)} |u_0(s) - m(u_0)|^2 ds,
\end{aligned}$$

where the conclusion follows. \square

Corollary 7.23. *Let $r_0 > 0$, $q \in \mathbb{R}^2$ be such that $r_0 < \frac{1}{2}d(q, \partial B(0, 1))$, suppose $u_0 \in C^1(\partial B(q, r_0) \cap \mathbb{R}^2, \mathbb{R})$, and denote by $m(u_0) = \frac{1}{2\pi r_0} \int_{\partial B(q, r_0)} u_0$ its average.*

Then for all $u \in C^1(\overline{B(0, 1)} \setminus B(q, r_0)) \cap \mathbb{R}^2, \mathbb{R})$ that satisfies

$$(7.24) \quad u|_{\partial B(q, r_0)} = u_0$$

we have

$$(7.25) \quad \int_{B(0, 1) \setminus B(q, r_0)} |\nabla u|^2 \geq \frac{1}{4} r_0^{-1} \int_{\partial B(q, r_0)} |u_0 - m(u_0)|^2.$$

Proof.

Set $R = d(q, \partial B(0, 1)) < 1$, then $r_0 < \frac{1}{2}R$ and $B(q, R) \subset B(0, 1)$. Thus we can apply Proposition 7.1 to $B(q, R) \setminus B(q, r_0)$ and by substituting $y = \frac{x-q}{R}$,

$$\begin{aligned}
(7.26) \quad \int_{B(q, R) \setminus B(q, r_0)} |\nabla_x u|^2 dx &= \int_{B(0, 1) \setminus B(0, \frac{r_0}{R})} \left| \frac{1}{R} \nabla_y u \right|^2 R^2 dy \\
&= \int_{B(0, 1) \setminus B(0, \frac{r_0}{R})} |\nabla_y u|^2 dy \geq \frac{1}{4} \left(\frac{R}{r_0} \right) \int_{\partial B(0, \frac{r_0}{R})} |u - m(u_0)|^2 dy
\end{aligned}$$

since $\frac{r_0}{R} < \frac{1}{2}$. But

$$(7.27) \quad \int_{\partial B(0, \frac{r_0}{R})} |u - m(u_0)|^2 dy = \int_{\partial B(q, r_0)} |u - m(u_0)|^2 \left(\frac{1}{R} \right) dx,$$

hence

$$(7.28) \quad \int_{B(q,R) \setminus B(q,r_0)} |\nabla_x u|^2 dx \geq \frac{1}{4} r_0^{-1} \int_{\partial B(q,r_0)} |u - m(u_0)|^2 dx.$$

Now since $B(q, R) \subset B(0, 1)$, we have

$$(7.29) \quad \int_{B(0,1) \setminus B(q,r_0)} |\nabla u|^2 \geq \frac{1}{4} r_0^{-1} \int_{\partial B(q,r_0)} |u - m(u_0)|^2 dx.$$

□

Lemma 7.30. *Let $0 < r_0 < 1$, and $u \in C^1(B(0,1) \setminus B(0,r_0), \mathbb{R})$ be such that $u|_{\partial B(0,r_0)} = \delta r_0$ and $u|_{\partial B(0,1)} = 0$; then we have*

$$(7.31) \quad \int_{B(0,1) \setminus B(0,r_0)} |\nabla u|^2 \geq \frac{2\pi\delta^2 r_0^2}{|\log r_0|}.$$

Proof.

Set $f(r, \theta) = A \log r$ with $A = \frac{\delta r_0}{\log r_0}$. Then f is the harmonic extension with the given boundary value, and

$$(7.32) \quad \frac{\partial f}{\partial r} = \frac{A}{r}, \quad \frac{\partial f}{\partial \theta} = 0.$$

Hence

$$(7.33) \quad |\nabla f|^2 = \left| \frac{\partial f}{\partial r} \right|^2 + \left| \frac{1}{r} \frac{\partial f}{\partial \theta} \right|^2 = \frac{A^2}{r^2}.$$

As a result

$$(7.34) \quad \begin{aligned} \int_{B(0,1) \setminus B(0,r_0)} |\nabla f|^2 &= \int_0^{2\pi} d\theta \int_{r_0}^1 r dr |\nabla f|^2 = 2\pi \int_{r_0}^1 r dr \frac{A^2}{r^2} \\ &= 2\pi A^2 |\log r_0| = \frac{2\pi\delta^2 r_0^2}{|\log r_0|} \end{aligned}$$

which gives

$$(7.35) \quad \int_{B(0,1) \setminus B(0,r_0)} |\nabla u|^2 \geq \frac{2\pi\delta^2 r_0^2}{|\log r_0|}$$

since f is harmonic. □

Corollary 7.36. *For all $0 < \epsilon < 1$, there exists $C = C(\epsilon) > 100$ such that if $0 < r_0 < 1$, $u \in C^1(B(0,1) \setminus B(0,r_0), \mathbb{R})$ and*

$$(7.37) \quad u|_{\partial B(0,r_0)} > \delta r_0 - \frac{\delta r_0}{C} \text{ and } u|_{\partial B(0,1)} < \frac{\delta r_0}{C}$$

then

$$(7.38) \quad \int_{B(0,1) \setminus B(0,r_0)} |\nabla u|^2 \geq \epsilon \frac{2\pi\delta^2 r_0^2}{|\log r_0|}.$$

Proof. We will use the following lemma:

Lemma 7.39. *Let $0 < r < 1$, let f, g be two harmonic functions on $B(0, 1) \setminus \overline{B(0, r)}$, with $g|_{\partial B(0, 1)} = a < b = g|_{\partial B(0, r)}$, and $f \leq g$ on $\partial B(0, 1)$, $f \geq g$ on $\partial B(0, r)$. Then*

$$(7.40) \quad \int_{B(0, 1) \setminus B(0, r)} |\nabla f|^2 \geq \int_{B(0, 1) \setminus B(0, r)} |\nabla g|^2.$$

Let us admit this lemma for the moment and use it to prove Corollary 7.36. For each C , set $r = r_0$, $f = u$, and g such that $g|_{\partial B(0, 1)} = \frac{\delta}{C} r_0$, $g|_{\partial B(0, r)} = (1 - \frac{1}{C})\delta r_0$. Then we get

$$(7.41) \quad \int_{B(0, 1) \setminus B(0, r_0)} |\nabla u|^2 \geq \int_{B(0, 1) \setminus B(0, r_0)} |\nabla g|^2 = (1 - \frac{2}{C})^2 \frac{2\pi\delta^2 r_0^2}{|\log r_0|}$$

and for each $\epsilon < 1$ we can find C large enough such that $(1 - \frac{2}{C})^2 \geq \epsilon$, thus complete the proof of Corollary 7.36.

Now let us prove Lemma 7.39. Set $h = (f - g)\nabla(f + g)$; then

$$(7.42) \quad \operatorname{div} h = |\nabla f|^2 - |\nabla g|^2$$

since $\Delta f = \Delta g = 0$. Denote by $U = B(0, 1) \setminus B(0, r)$, then by Stokes formula:

$$(7.43) \quad \begin{aligned} \int_U |\nabla f|^2 - |\nabla g|^2 &= \int_U \operatorname{div} h = \int_{\partial U} h \cdot \vec{n} = \int_{\partial U} (f - g) \frac{\partial}{\partial \vec{n}} (f + g) \\ &= \int_{\partial U} (f - g) \frac{\partial}{\partial \vec{n}} (f - g) + 2 \int_{\partial U} (f - g) \frac{\partial}{\partial \vec{n}} g, \end{aligned}$$

where \vec{n} is exterior unit normal vector.

For the first term, since $k = f - g$ is harmonic, by Green's formula

$$(7.44) \quad \int_{\partial U} k \frac{\partial}{\partial \vec{n}} k = \int_U (\nabla k \cdot \nabla k) + \int_U (k \Delta k) = \int_U |\nabla k|^2 \geq 0.$$

For the second, by the boundary condition, on $\partial B(0, 1)$, $f - g \leq 0$ and $\frac{\partial}{\partial \vec{n}} g < 0$, hence $(f - g) \frac{\partial}{\partial \vec{n}} g \geq 0$; similar for $\partial B(0, r)$, thus we have

$$(7.45) \quad \int_{\partial U} (f - g) \frac{\partial}{\partial \vec{n}} g \geq 0.$$

As a result

$$(7.46) \quad \int_U |\nabla f|^2 - |\nabla g|^2 \geq 0.$$

□

8 Conclusion

After all the preparations in the previous sections, we are going to conclude for Theorem 1.2 in this section.

So fix a $\epsilon < \epsilon_0$. We are going to estimate the Hausdorff measure of E_k for k large enough. For each k fixed, we have chosen o_k and r_k as in Proposition 5.11. Then by Proposition 6.1(1), $E_k \cap D_k(0, \frac{39}{40}) \setminus D_k(o_k, \frac{1}{10}r_k)$ is composed of two disjoint pieces $G_k, i = 1, 2$ such that (6.2) and (6.3) hold, where we replace q_n, s_n by o_k, r_k . Moreover we can also suppose that $r_k < 2^{-5}$, since k is large.

Proposition 8.1. *For all $\epsilon > 0$, there exists $0 < \delta = \delta(\epsilon) < \epsilon$ and $\theta_0 = \theta_0(\epsilon) < \frac{\pi}{2}$, that depend only on ϵ , and satisfy the following properties. If $\frac{\pi}{2} > \theta > \theta_0$ (which means $\theta = (\theta_1, \theta_2)$ with $\theta_0 < \theta_1 \leq \theta_2 < \frac{\pi}{2}$, $i = 1, 2$) and E is minimal in $B(0, 1)$ which is also δ near $P_\theta = P_\theta^1 \cup_\theta P_\theta^2$ in $B(0, 1) \setminus B(0, \frac{1}{2})$, and if moreover*

$$(8.2) \quad p_0^i(E) \supset P_0^i \cap B(0, \frac{3}{4})$$

where p_0^i denotes the orthogonal projection on $P_0^i, i = 1, 2$, then E is ϵ near P_θ in $B(0, 1)$.

Proof.

We prove it by contradiction. So suppose the proposition is not true. Then there exists $\epsilon > 0$, two sequences $\delta_l \rightarrow 0$ and $\theta_l \rightarrow (\frac{\pi}{2}, \frac{\pi}{2})$, and a sequence of minimal sets E_l in $B(0, 1)$ such that E_l is δ_l near $P_l = P_{\theta_l}$ in $B(0, 1) \setminus B(0, \frac{1}{2})$, and

$$(8.3) \quad p_0^i(E_l) \supset P_0^i \cap B(0, \frac{3}{4}),$$

but E_l is not ϵ near P_l in $B(0, 1)$.

Since P_l converges to $P_0 = P_0^1 \cup_\perp P_0^2$, $\delta_l \rightarrow 0$ and ϵ is fixed, there exists a sequence $\{a_l\}$ which converges to 0, such that E_l is a_l near P_0 in $B(0, 1) \setminus B(0, \frac{1}{2})$, but not $\frac{\epsilon}{2}$ near P_0 in $B(0, 1)$ (because E_l is not ϵ near P_l , and P_l is $\frac{\epsilon}{2}$ near P_0 when l is large.)

Modulo extracting a subsequence, we can suppose that $\{E_l\}$ converges to a limit E_∞ . Then $E_\infty \cap B(0, 1) \setminus B(0, \frac{1}{2}) = P_0 \cap B(0, 1) \setminus B(0, \frac{1}{2})$.

We want to prove that

$$(8.4) \quad H^2(E_l \cap D(0, \frac{3}{4})) < \frac{9}{8}\pi + b_l, \text{ with } b_l \rightarrow 0 \text{ when } l \rightarrow \infty,$$

where $D(x, r)$ denotes $D_0(x, r)$ for short.

In fact, since E_l is very near P_0 in $B(0, 1) \setminus B(0, \frac{1}{2})$ when l is large, by the C^1 regularity of minimal sets (c.f. Thm 1.35), we know that $E_l \cap \partial D(0, \frac{3}{4}) = \Gamma_l^1 \cup \Gamma_l^2$, where Γ_l^1 and Γ_l^2 are two disjoint curves, Γ_l^i is the graph of a C^1 function $h_l^i : P_0^i \cap \partial D(0, \frac{3}{4}) \rightarrow P_0^{i\perp}$, with

$$(8.5) \quad \|h_l^i\|_\infty \rightarrow 0, k \rightarrow \infty \text{ et } \|\frac{d}{dx}h_l^i\|_\infty \leq 1, \forall l.$$

Now set $D_l^i = (P_0^i \cap D(0, \frac{3}{4})) \cup A_l^i$, where $A_l^i = \{(x, y) : x \in P_0^i \cap \partial D(0, \frac{3}{4}), y \in [0, h_l^i(x)]\}$, which is just a 2-dimensional thin surface between $P_0^i \cap \partial D(0, \frac{3}{4})$ and Γ_l^i , the graph of h_l^i . We can also write

$$(8.6) \quad A_l^i = \{(x, th_l^i(x)) : x \in P_0^i \cap \partial D(0, \frac{3}{4}), t \in [0, 1]\}.$$

It is not hard to see that D_l^i is a surface whose boundary is Γ_l^i , and $D_l = D_l^1 \cup D_l^2$ contains a deformation of E_l in $\overline{D}(0, \frac{3}{4})$.

Let us verify that

$$(8.7) \quad H^2(A_l^i) \leq \frac{3\sqrt{2}\pi}{2} \|h_l^i\|_\infty.$$

Since h_l^i is \mathbb{R}^2 -valued, the simplest is to use a parameterization. Set $g_l^i(x, t) : \partial D(0, \frac{3}{4}) \times [0, 1] \rightarrow \mathbb{R}^4$; $g_l^i(x, t) = (x, th_l^i(x))$, then A_l^i is its image. And

$$(8.8) \quad \frac{\partial}{\partial x} g_l^i = (1, t \frac{d}{dx} h_l^i(x)) ; \quad \frac{\partial}{\partial t} g_l^i = (0, h_l^i(x))$$

Therefore

$$(8.9) \quad |\overrightarrow{\frac{\partial}{\partial x} g_l^i} \times \overrightarrow{\frac{\partial}{\partial t} g_l^i}| = \sqrt{(1+t^2|\frac{\partial}{\partial x} h_l^i|^2)(|h_l^i|^2) - (t\overrightarrow{\frac{\partial}{\partial x} h_l^i} \cdot \overrightarrow{h_l^i})^2}.$$

But we know that $|\frac{\partial}{\partial x} h_l^i| \leq 1$, hence

$$(8.10) \quad |\overrightarrow{\frac{\partial}{\partial x} g_l^i} \times \overrightarrow{\frac{\partial}{\partial t} g_l^i}| \leq \sqrt{(1+t^2)|h_l^i|^2} \leq \sqrt{2}|h_l^i|,$$

and therefore

$$(8.11) \quad \begin{aligned} H^2(A_l^i) &= \int_{\partial D(0, \frac{3}{4}) \times [0, 1]} |\overrightarrow{\frac{\partial}{\partial x} g_l^i} \times \overrightarrow{\frac{\partial}{\partial t} g_l^i}| dt dx \\ &\leq \sqrt{2} \int_{\partial D(0, \frac{3}{4})} |h_l^i| \leq \frac{3\sqrt{2}}{2} \pi \|h_l^i\|_\infty. \end{aligned}$$

Thus we obtain

$$(8.12) \quad H^2(D_l^i) \leq \frac{9}{16} \pi + \frac{3\sqrt{2}}{2} \pi \|h_l^i\|_\infty$$

and

$$(8.13) \quad H^2(D_l) \leq \frac{9}{8} \pi + \frac{3\sqrt{2}}{2} \pi (\|h_l^1\|_\infty + \|h_l^2\|_\infty).$$

But E_l is locally minimal, hence

$$(8.14) \quad H^2(E_l \cap D(0, \frac{1}{2})) \leq H^2(D_l) = \frac{9}{8} \pi + \frac{3\sqrt{2}}{2} \pi (\|h_l^1\|_\infty + \|h_l^2\|_\infty).$$

Take $b_l = \frac{3\sqrt{2}}{2}\pi(\|h_l^1\|_\infty + \|h_l^2\|_\infty)$, thus we get (8.4), since $\|h_l^1\|_\infty + \|h_l^2\|_\infty$ converges to 0 when $l \rightarrow \infty$.

On the other hand, E_l converges to P_0 in $B(0, 1) \setminus B(0, \frac{1}{2})$, so we have

$$(8.15) \quad E_l \cap (B(0, 1) \setminus B(0, \frac{1}{2})) \rightarrow P_0 \cap (B(0, 1) \setminus B(0, \frac{1}{2})) = E_\infty \cap (B(0, 1) \setminus B(0, \frac{1}{2})).$$

By the lower semi continuity of Hausdorff measures for minimal sets ([4] Thm 3.4, and recall that E_∞ is the limit of E_l)

$$(8.16) \quad H^2(E_\infty \cap D(0, \frac{3}{4})) \leq \liminf_{k \rightarrow \infty} H^2(E_l \cap D(0, \frac{3}{4})).$$

So we have

$$(8.17) \quad \begin{aligned} H^2(E_\infty) &= H^2(E_\infty \cap (B(0, 1) \setminus D(0, \frac{3}{4}))) + H^2(E_\infty \cap D(0, \frac{3}{4})) \\ &\leq H^2(P_0 \cap (B(0, 1) \setminus D(0, \frac{3}{4}))) + \liminf_{k \rightarrow \infty} H^2(E_l \cap D(0, \frac{3}{4})) \\ &= 2\pi + \liminf_{k \rightarrow \infty} b_l = 2\pi. \end{aligned}$$

Now by (8.3) and the fact that E_∞ is the limit of E_l , we know that

$$(8.18) \quad p_0^i(E_\infty) \supset P_0^i \cap B(0, \frac{3}{4}).$$

By hypothesis, E_l converges to P_0 in $B(0, 1) \setminus B(0, \frac{1}{2})$, hence $E_\infty \cap B(0, 1) \setminus B(0, \frac{1}{2}) = P_0 \cap B(0, 1) \setminus B(0, \frac{1}{2})$, and therefore

$$(8.19) \quad p_0^i(E_\infty) \supset p_0^i(E_\infty \cap B(0, 1) \setminus B(0, \frac{1}{2})) = P_0^i \cap B(0, 1) \setminus B(0, \frac{1}{2}).$$

Thus we have

$$(8.20) \quad p_0^i(E_\infty) \supset P_0^i \cap B(0, 1),$$

and

$$(8.21) \quad E_\infty \cap \partial B(0, 1) = P_0 \cap \partial B(0, 1)$$

because E_l converges to P_0 in $B(0, 1) \setminus B(0, \frac{1}{2})$. Then by Theorem 3.1, (8.17), (8.20) and (8.21) give that

$$(8.22) \quad E_\infty = P_0.$$

This is impossible, because E_l is $\frac{\epsilon}{2}$ far from P_0 . Thus we complete the proof of Proposition 8.1. \square

Now for $0 < \theta = (\theta_1, \theta_2)$ with $0 < \theta_1 \leq \theta_2 < \frac{\pi}{2}$, set, for each $x \in \mathbb{R}^4, r > 0$,

$$(8.23) \quad D_\theta(x, r) = x + \{p_\theta^{1-1}[B(0, r) \cap P_\theta^1] \cap p_\theta^{2-1}[B(0, r) \cap P_\theta^2]\},$$

where $P_\theta = P_\theta^1 \cup P_\theta^2$ is the union of two planes with characteristic angles $\theta_1 \leq \theta_2$, and denote by p_θ^i the orthogonal projection to $P_\theta^i, i = 1, 2$. Then we have

Corollary 8.24. *For all $\epsilon > 0$, there exists $0 < \delta < \epsilon$ and $0 < \theta_0 < \frac{\pi}{2}$, which do not depend on ϵ , with the following properties. If $\theta_0 < \theta < \frac{\pi}{2}$, and if E is minimal in $D_\theta(0, 1)$ and is δ near P_θ in $D_\theta(0, 1) \setminus D_\theta(0, \frac{1}{4})$, and moreover*

$$(8.25) \quad p_\theta^i(E) \supset P_\theta^i \cap B(0, \frac{3}{4}),$$

then E is ϵ near P_θ in $D_\theta(0, 1)$.

Proof. First observe that there exists $0 < \phi < \frac{\pi}{2}$ such that for all $0 < \theta < \phi$ we have

$$(8.26) \quad B(x, r) \subset D_\theta(x, r) \subset B(x, 2r).$$

Then for $\epsilon > 0$, take $\delta = \delta(\epsilon)$ and $\theta_0 = \max\{\phi, \theta_0(\epsilon)\}$, where $\theta_0(\epsilon)$ and $\delta(\epsilon)$ are as in Proposition 8.1. Then if E is δ near P_θ in $D_\theta(0, 1) \setminus D_\theta(0, \frac{1}{4})$, it is δ near P_θ in $B(0, 1) \setminus B(0, \frac{1}{2})$. By Proposition 8.1, E is ϵ near P_θ in $B(0, 1)$. Hence E is $\epsilon = \max\{\delta, \epsilon\}$ near P_θ in $B(0, 1) \cup [D_\theta(0, 1) \setminus D_\theta(0, \frac{1}{4})] = D_\theta(0, 1)$. \square

Proof of Theorem 1.2. Take all the notation at the beginning of this section.

Fix a k large, and denote by $D(x, r) = D_k(x, r)$, $C^i(x, r) = C_k^i(x, r)$ for $i = 1, 2$, and $d_{x,r} = d_{x,r}^k$.

We know that in $D(o_k, r_k)$, E_k is not ϵr_k near any translation of P_k , so by Corollary 8.24, E_k is not δr_k near any translation of P_k in $D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k)$. However by (6.2), $E_k \cap D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k) = [G^1 \cup G^2] \cap [D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k)]$, where G^i is a C^1 graph of $P_k^i \cap D(0, \frac{39}{40}) \setminus D(o_k, \frac{1}{10}r_k)$. Hence there exists $i \in \{1, 2\}$ such that in $D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k)$, G^i is not δr_k near any translation of P_k^i . Without loss of generality, we can suppose this is the case for $i = 1$.

Then denote by $P = P_k^1$ for short, and let g^1 be as in (6.2); then g^1 is a map from P to P^\perp , and is therefore from \mathbb{R}^2 to \mathbb{R}^2 . Write $g^1 = (\varphi_1, \varphi_2)$, where $\varphi_i : \mathbb{R}^2 \rightarrow \mathbb{R}$. Then since the graph of g^1 is δr_k far from all translation of P , there exists $j \in \{1, 2\}$ such that

$$(8.27) \quad \sup_{x, y \in P \cap D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k)} |\varphi_j(x) - \varphi_j(y)| \geq \frac{1}{2}r_k\delta.$$

Suppose this is true for $j = 1$. Denote by

$$(8.28) \quad K = \{(z, \varphi_1(z)) : z \in (D(0, \frac{3}{4}) \setminus D(o_k, \frac{1}{4}r_k)) \cap P\},$$

then

$$(8.29) \quad \begin{aligned} K \text{ is the orthogonal projection of } G^1 \cap D(0, \frac{3}{4}) \\ \text{on a 3-dimensional subspace of } \mathbb{R}^4. \end{aligned}$$

For $\frac{1}{4}r_k \leq s \leq r_k$, define

$$(8.30) \quad \Gamma_s = K \cap p^{-1}(\partial D(o_k, s) \cap P) = \{(x, \varphi_1(x)) | x \in \partial D(o_k, s) \cap P\}$$

the graph of φ_1 on $\partial D(o_k, s) \cap P$.

We know that the graph of φ_1 is $\frac{1}{2}\delta_k$ far from P in $D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k)$; then there are two cases:

1st case: there exists $t \in [\frac{1}{4}r_k, r_k]$ such that

$$(8.31) \quad \sup_{x, y \in \Gamma_t} \{|\varphi_1(x) - \varphi_1(y)|\} \geq \frac{\delta}{C}r_k,$$

where $C = 4C(\frac{1}{2})$ is the constant of Corollary 7.36.

Then there exists $a, b \in \Gamma_t$ such that $|\varphi_1(a) - \varphi_1(b)| > \frac{\delta}{C}r_k \geq \frac{\delta}{C}t$. Since $\|\nabla \varphi_1\|_\infty \leq \|\nabla \varphi\|_\infty < 1$, we have

$$(8.32) \quad \int_{\Gamma_t} |\varphi_1 - m(\varphi_1)|^2 \geq \frac{t^3 \delta^3}{4C^3} = (\frac{4}{3}t\delta)^3 (\frac{27}{4^4 C^3}).$$

Now in $D(0, \frac{3}{4})$ we have $d(0, o_k) < 6\epsilon \leq 10\epsilon \cdot \frac{3}{4}$, and $s < r_k < \frac{1}{8} < \frac{1}{2} \times \frac{3}{4}$, therefore we can apply Corollary 7.23 and obtain

$$(8.33) \quad \int_{(D(0, \frac{3}{4}) \setminus D(o_k, t)) \cap P} |\nabla \varphi_1|^2 \geq C_1(\delta)t^2.$$

2nd case: for all $\frac{1}{4}r_k \leq s \leq r_k$,

$$(8.34) \quad \sup_{x, y \in \Gamma_s} \{|\varphi_1(x) - \varphi_1(y)|\} \leq \frac{\delta}{C}r_k.$$

However, since

$$(8.35) \quad \begin{aligned} \frac{1}{2}r_k\delta &\leq \sup\{|\varphi_1(x) - \varphi_2(y)| : x, y \in P \cap D(o_k, r_k) \setminus D(o_k, \frac{1}{4}r_k)\} \\ &= \sup\{|\varphi_1(x) - \varphi_2(y)| : s, s' \in [\frac{1}{4}r_k, r_k], x \in \Gamma_s, y \in \Gamma_{s'}\}, \end{aligned}$$

there existe $\frac{1}{4}r_k \leq t < t' \leq r_k$ such that

$$(8.36) \quad \sup_{x \in \Gamma_t, y \in \Gamma_{t'}} \{|\varphi_1(x) - \varphi_1(y)|\} \geq \frac{1}{2}r_k\delta.$$

Fix t and t' , and without loss of generality, suppose that

$$(8.37) \quad \sup_{x \in \Gamma_t, y \in \Gamma_{t'}} \{\varphi_1(x) - \varphi_1(y)\} \geq \frac{1}{2}r_k\delta.$$

Then

$$(8.38) \quad \inf_{x \in \Gamma_t} \varphi_1(x) - \sup_{x \in \Gamma_{t'}} \varphi_1(x) \geq \frac{1}{2}r_k\delta - 2\frac{\delta}{C}r_k = (1 - \frac{2}{C(\frac{1}{2})})\frac{\delta}{2}r_k \geq (1 - \frac{2}{C(\frac{1}{2})})\frac{\delta}{2}t'$$

because $C = 4C(\frac{1}{2})$.

Now look at what happens in the ball $D(o_k, t') \cap P$. Apply Corollary 7.36 to the scale t' , we get

$$(8.39) \quad \int_{(D(o_k, t') \setminus D(o_k, t)) \cap P} |\nabla \varphi_1|^2 \geq C(\delta, \frac{1}{2}) \frac{\pi(\frac{\delta}{2})^2 t'^2}{\log \frac{t'}{t}}.$$

Then since $\frac{t'}{t} \leq 4, t' > t$, we have

$$(8.40) \quad \int_{((D(o_k, t') \setminus D(o_k, t)) \cap P} |\nabla \varphi_1|^2 \geq C_2(\delta) t^2.$$

In both cases we pose $t_k = t$. The discussion above yields that there exists a constant $C_0(\delta) = \min\{C_1(\delta), C_2(\delta)\}$, which depends only on δ , such that

$$(8.41) \quad \int_{(D(0, \frac{3}{4}) \setminus D(o_k, t_k)) \cap P} |\nabla \varphi_1|^2 \geq C_0(\delta) t_k^2.$$

On the other hand, since $|\nabla \varphi_1| \leq |\nabla g^1| < 1$,

$$(8.42) \quad \sqrt{1 + |\nabla \varphi_1|^2} > \sqrt{1 + \frac{1}{2} |\nabla \varphi_1|^2 + \frac{1}{16} |\nabla \varphi_1|^4} = 1 + \frac{1}{4} |\nabla \varphi_1|^2.$$

Hence

$$(8.43) \quad \begin{aligned} H^2(K \setminus C_k^1(o_k, t_k)) &= \int_{D(0, \frac{3}{4}) \setminus C_k^1(o_k, t_k) \cap P} \sqrt{1 + |\nabla \varphi_1|^2} \geq \int_{D(0, \frac{3}{4}) \setminus C_k^1(o_k, t_k) \cap P} 1 + \frac{1}{4} |\nabla \varphi_1|^2 \\ &\geq H^2((D(0, \frac{3}{4}) \setminus C_k^1(o_k, t_k)) \cap P) + \frac{1}{4} \int_{D(0, \frac{3}{4}) \setminus C_k^1(o_k, t_k) \cap P} |\nabla \varphi_1|^2 \\ &= H^2((D(0, \frac{3}{4}) \setminus C_k^1(o_k, t_k)) \cap P_k^1) + C_0(\delta) t_k^2. \end{aligned}$$

Then by (8.29) we get

$$(8.44) \quad \begin{aligned} H^2(G^1 \cap D(0, \frac{3}{4}) \setminus D(o_k, t_k)) &\geq H^2(K \setminus D(o_k, t_k)) \\ &\geq H^2((P_k^1 + o_k) \cap D(0, \frac{3}{4}) \setminus D(o_k, t_k)) + C(\delta) t_k^2 \\ &= H^2(P_k^1 \cap D(0, \frac{3}{4}) \setminus D(0, t_k)) + C_0(\delta) t_k^2. \end{aligned}$$

Thus we obtain an estimate for the regular part of E_k . Next for all other parts of E_k we will control their measures by projection.

So let us decompose E_k . Set $F_1 = E_k \cap D(o_k, t_k)$, $F_2 = G_{t_k}^2$, $F_3 = G_{t_k}^1 \setminus D(0, \frac{3}{4})$, and $F_4 = G_{t_k}^1 \cap D(0, \frac{3}{4})$, where G_t^i is defined as in Proposition 6.1(2). Then F_i are disjoint.

For F_1 , by Proposition 2.19 and Lemma 2.27

$$(8.45) \quad (1 + 2 \cos \theta_k(1)) H^2(F_1) \geq H^2(p_k^1(F_1)) + H^2(p_k^2(F_1)),$$

where $\theta_k = (\theta_k(1), \theta_k(2))$ with $\theta_k(1) \leq \theta_k(2)$. (Recall that E_k has the same boundary as $P_k = P^1 \cup_{\theta_k} P^2$ with $\theta_k \geq \frac{\pi}{2} - \frac{1}{k}$). However since $\theta_k \geq \frac{\pi}{2} - \frac{1}{k}$, we have

$$(8.46) \quad H^2(F_1) \geq (1 - \frac{3}{k}) [H^2(p_k^1(F_1)) + H^2(p_k^2(F_1))]$$

when k is large. On the other hand, by Proposition 6.1(4),

$$(8.47) \quad p_k^i(F_1) \supset P_k^i \cap C_k^i(o_k, t_k).$$

As a result

$$\begin{aligned}
(8.48) \quad H^2(F_1) &\geq (1 - \frac{3}{k})H^2((P_k + o_k) \cap D(o_k, t_k)) \\
&\geq H^2((P_k + o_k) \cap D(o_k, t_k)) - \frac{C}{k}t_k^2 = H^2(P_k \cap D(0, t_k)) - \frac{C}{k}t_k^2.
\end{aligned}$$

For F_2 , by Proposition 6.1(2), we have

$$(8.49) \quad p_k^2(F_2) = p_k^2(G_{t_k}^2) \supset P_k^i \cap D(0, 1) \setminus C_k^2(o_k, t_k).$$

Hence

$$(8.50) \quad H^2(F_2) \geq H^2[P_k^2 \cap D(0, 1) \setminus C_k^2(o_k, t_k)] = H^2[P_k^2 \cap D(0, 1) \setminus D(0, t_k)].$$

For F_3 , still by Proposition 6.1(2), and by the definition of F_3 , we have

$$(8.51) \quad p_k^1(F_3) \supset p_k^1(G_{t_k}^1) \setminus p_k^1(D(0, \frac{3}{4})) \supset P_k^1 \cap D(0, 1) \setminus D(0, \frac{3}{4}).$$

Hence

$$(8.52) \quad H^2(F_3) \geq H^2(P_k^1 \cap D(0, 1) \setminus D(0, \frac{3}{4})).$$

For the last part, the definition of F_4 gives

$$(8.53) \quad H^2(F_4) = H^2(G^1 \setminus D(o_k, t_k)) \geq H^2(P_k^1 \cap D(0, \frac{3}{4}) \setminus D(0, t_k)) + C_0(\delta)t_k^2.$$

Now we add measures of these four pieces together and get

$$\begin{aligned}
(8.54) \quad H^2(E_k) &= H^2(F_1) + H^2(F_2) + H^2(F_3) + H^2(F_4) \\
&\geq H^2(P_k \cap D(0, 1)) + t_k^2(C_0(\delta) - \frac{C}{k}).
\end{aligned}$$

Then when k is such that $C(\delta) > \frac{C}{k}$, we have

$$(8.55) \quad H^2(E_k) > H^2(P_k \cap D(0, 1)),$$

which contradicts Proposition 4.8(4). Thus the proof of Theorem 1.2 is completed. \square

As a final remark, we give two similar theorems below.

Theorem 8.56 (minimality of the union of n almost orthogonal m -dimensional planes). *For each $m \geq 2$ and $n \geq 2$, there exists $0 < \theta < \frac{\pi}{2}$, such that if P^1, P^2, \dots, P^n are n planes of dimension m in \mathbb{R}^{nm} with characteristic angles $\alpha^{ij} = (\alpha_1^{ij}, \alpha_2^{ij}, \dots, \alpha_m^{ij})$ between P^i and P^j , $1 \leq i < j \leq n$, which verify $\theta < \alpha_1^{ij} \leq \alpha_2^{ij} \leq \dots \leq \alpha_m^{ij} \leq \frac{\pi}{2}$ for all $1 \leq i < j \leq n$, then their union $\cup_{i=1}^n P^i$ is a minimal cone.*

Theorem 8.57 (minimality of the union of a plane and a \mathbb{Y} set which are almost orthogonal). *There exists $0 < \theta < \frac{\pi}{2}$, such that if P and Q are two subspaces of \mathbb{R}^5 of dimension 2 and 3 respectively, which verify*

$$(8.58) \quad \text{for all unit simple vectors } u \in P, v \in Q, \text{ then the angle between } u \text{ and } v \text{ is larger than } \theta,$$

and if Y is a \mathbb{Y} set in Q centered at the point of $P \cap Q$, then the union $P \cup Y$ is a minimal cone.

The general idea for the proofs of the two theorems are somehow similar to the proof of Theorem 1.2. But there are also some non trivial modifications, especially for Theorem 8.57. See [13] for more detail of the proofs. We could have continued to discuss one by one that if the almost orthogonal unions of other pairs or families of minimal cones are minimal. In fact Theorem 8.57 was the first try, where we replaced one plane by the simplest minimal cone \mathbb{Y} . But then the author noticed that the proof became already a bit too complicated, so she stopped at this step.

References

- [1] William K. Allard. On the first variation of a varifold. *Ann.of Math.(2)*, 95:417–491, 1972.
- [2] F. J. Almgren. *Plateau’s problem: An invitation to varifold geometry*. W.A. Benjamin, 1966.
- [3] F. J. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Memoirs of the American Mathematical Society*, 4(165), 1976.
- [4] Guy David. Limits of Almgren-quasiminimal sets. *Proceedings of the conference on Harmonic Analysis, Mount Holyoke, A.M.S. Contemporary Mathematics series*, 320:119–145, 2003.
- [5] Guy David. Hölder regularity of two-dimensional almost-minimal sets in \mathbb{R}^n . *Annales de la Faculté des Sciences de Toulouse*, XVIII(1):65–246, 2009.
- [6] Guy David. $C^{1+\alpha}$ -regularity for two-dimensional almost-minimal sets in \mathbb{R}^n . *Journal of geometric analysis*, 20(4):837–954, 2010.
- [7] Guy David and Stephen Semmes. Uniform rectifiability and quasiminimizing sets of arbitrary codimension. *Memoirs of the A.M.S.*, 144(687), 2000.
- [8] Herbert Federer. *Geometric measure theory*. Grundlehren der Mathematischen Wissenschaften 153. Springer Verlag, 1969.
- [9] Vincent Feuvrier. *Un résultat d’existence pour les ensembles minimaux par optimisation sur des grilles polyédrales*. PhD thesis, Université de Paris-Sud 11, orsay, september 2008.

- [10] A. Heppes. Isogonal sphärischen Netze. *Ann.Univ.Sci.Budapest Eötvös Sect.Math*, 7:41–48, 1964.
- [11] E. Lamarle. Sur la stabilité des systèmes liquides en lames minces. *Mémoires de l'Académie Royale de Belgique*, 35:3–104, 1864.
- [12] Gary Lawlor. Pairs of planes which are not size-minimizing. *Indiana Univ. Math. J.*, 43:651–661, 1994.
- [13] Xiangyu Liang. *Ensembles et cônes minimaux de dimension 2 dans les espaces euclidiens*. PhD thesis, Université de Paris-Sud 11, Orsay, December 2010.
- [14] Xiangyu Liang. Topological minimal sets and their applications. *preprint, Orsay, arXiv:1103.3871*, 2011.
- [15] Pertti Mattila. *Geometry of sets and measures in Euclidean space*. Cambridge Studies in Advanced Mathematics 44. Cambridge University Press, 1995.
- [16] Frank Morgan. Soap films and mathematics. *Proceedings of Symposia in Pure Mathematics*, 54:Part 1, 1993.
- [17] Thierry De Pauw. Size minimising surfaces. *Annales scientifiques de l'école normale supérieure*, 42(4):37–101, 2009.
- [18] Thierry De Pauw and Robert Hardt. Size minimization and approximating problems. *Calculus of Variations*, 17:405–442, 2003.
- [19] Walter Rudin. *Real and complex analysis*. McGRAW-HILL Publishins Co., 3rd edition, 1987.
- [20] Jean Taylor. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. of Math.(2)*, 103:489–539, 1976.